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Papers offered for publication in the March 1960 issue should reach the Assistant Secretary of the Society by mid-November 1959.

Volume 2 No. 4 December 1959

The following contributions have been accepted for publication in the above issue:

K. M. Creer, A.C. demagnetization of unstable Triassic Keuper marls from S.W. England.

T. Rikitake, Anomaly of geomagnetic variations in Japan.

K. F. Bowden, L. A. Fairbairn & D. Hughes, The distribution of shearing stresses in a tidal current.

K. M. Creer, E. Irving & A. E. M. Nairn, Palaeomagnetism of the Great Whin Sill.

A. L. Hales & D. I. Gough, Gravity anomalies and crustal structure in South Africa.

J. Macdowell, A note on the relation between wind velocity and the amplitude of microseisms on the coast of Antarctica.

J. E. Jackson, Observations with the Cambridge pendulum apparatus in North, Central and South America in 1958.

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The Transmission of Geomagnetic Disturbances through the Atmosphere and Interplanetary Space

J. H. Piddington

(Received 1959 February 23)

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(1) Up to a few hundred kilometres the medium behaves, for waves of all periods between 1 s and 10^3 s, *as a rigid conductor* and as a dispersive medium.

(2) Above about 10^3 km the disturbances travel as hydromagnetic waves in the ion plasma alone. Losses are small in this region but transmission is likely to be complicated by anisotropic transmission of the *O* wave and by refraction and partial reflection.

(3) The currents responsible for all observed geomagnetic disturbances must flow at levels below about 1 000 km.

(4) The problem of the penetration of solar gas into the Earth's field is discussed.

(5) Some earlier theories of the main phase of an SC storm are discussed and an alternative suggested, whereby some lines of force of the Earth's field are carried away along the Sun–Earth line.

(6) Some properties of micropulsations are explained in terms of the transmission of hydromagnetic waves.

(7) The "effective" conductivity (σ_3) of the atmosphere out to several Earth radii has its maximum about 100 km, above which it rapidly falls to about 10^{-16} e.m.u. or less and does not rise again.

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1. Introduction

Geomagnetic disturbances observed near the Earth's surface have fluctuation periods ranging from a fraction of a second in the case of some micropulsations up to some days for the slowest storm variations.

It is generally accepted that some of these disturbances originate outside the Earth's atmosphere. They must comprise, then, electromagnetic waves which propagate through the Earth's atmosphere and perhaps through interplanetary space. In so propagating they will suffer phase delay, refraction, absorption and partial reflection and the disturbance finally observed at the Earth's surface will depend on all these factors as well as on the nature of the original disturbance.

Propagation through the ionosphere has been considered by Ashour & Price (1948) and others in a theory in which the medium does not move under the influence of the electromagnetic field. A number of workers have considered the passage of micropulsations as downward travelling hydromagnetic waves. As will be seen below, these theories do not take full account of the existence of two gas components, an ion-electron plasma (here called the plasma) and neutral atoms. These gases together constitute the whole, partially ionized, gas of the atmosphere and interplanetary space. Under the influence of an electromagnetic field they move to some extent, independently.

The transmission of geomagnetic disturbances is a problem in hydromagnetics in a gas mixture which may not be treated as a *single fluid*. It is found that the concept of electrical conductivity of the whole gas is not valid and that the medium is *dispersive* for hydromagnetic waves.

The theory developed here must be regarded as a preliminary step. It concerns weak waves in a uniform medium and so excludes possibly important refraction and reflection effects. Nevertheless, it may reveal some significant transmission effects and throw some light on the difficult theory of the origins of the geomagnetic disturbances.

2. The transmission equations

The transmission of hydromagnetic waves through a partially ionized gas may be studied by using Maxwell's field equations, together with the separate equations of motion of the *three* gas components present (for application to micropulsations and references to earlier work, see Akasofu 1956). This approach is of course valid, but adds greatly and unnecessarily to the complexity of the problem. The reason is that Maxwell's displacement currents may be neglected in these problems and so, to the same order of accuracy, we may write $\mathrm{div}\,\mathbf{j} = 0$, where $\mathbf{j}$ is the electric current density.* This means that the electron gas and the heavy ion gas may be considered a single dynamic entity and have a single force-momentum equation (for a previous discussion see Piddington 1954).

The situation is much simplified by using the equations of motion of *two* gases only: the plasma and the neutral atom gas. It is further simplified by neglecting ordinary viscosity due to thermal motions of the particles; this is usually unimportant compared with "neutral atom damping" (Piddington 1957). Finally, the effects of gas pressure gradients caused by the wave are neglected. These are not present when the waves are propagated along the magnetic field; for transverse propagation compression effects are small provided the velocity of sound is small compared with the hydromagnetic velocity V (Piddington 1955), which is the case, except for waves of long period in ionospheric E region.

* This does not mean that electric charge may not accumulate on boundaries, but that the currents involved in causing such accumulations have negligible dynamic effects.

The hydromagnetic equations may now be listed (Piddington 1956):

$$\nabla^2\mathbf{H} = 4\pi\sigma_3\left\{\frac{\partial\mathbf{H}}{\partial t} - \operatorname{curl}(\mathbf{v}\wedge\mathbf{H})\right\} + \frac{\sigma_2}{\sigma_1}\frac{\partial}{\partial z}\operatorname{curl}\mathbf{H}, \tag{1}$$

$$\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}-\mathbf{v}')\frac{\eta}{\tau} + \frac{V^2}{\mathbf{H}^2}(\mathbf{H}\wedge\operatorname{curl}\mathbf{H}) = 0, \tag{2}$$

$$\frac{\partial v'}{\partial t} = (v-v')1/\tau, \tag{3}$$

where

$$V = H_0(4\pi\rho)^{-\frac{1}{2}} \tag{4}$$

Here σ_1, σ_2, σ_3 are the well-known conductivity components (e.m.u.) of the plasma alone and $\mathbf{H}$ the magnetic field, with a steady component $\mathbf{H}_0$ directed along O.z The velocity vectors of the plasma and neutral atom gas are $\mathbf{v}$ and $\mathbf{v}'$ and $\eta = \rho_n/\rho$ is the ratio of the mass densities of neutral atoms and ions. Finally τ is the period taken to accelerate the neutral atoms to the same velocity as the plasma, assuming the latter remains constant during this period (see Piddington 1957, Appendix I).

Although we are dealing with a partially ionized gas, equation (1) refers to a fully ionized gas because the effects of neutral atoms are introduced separately by equations (2) and (3). This method is essential when dealing with waves of frequency greater than the neutral atom collision frequency, when the concept of conductivity in the partially ionized gas is not valid. This applies for most geomagnetic disturbances.

Equation (1) was derived with the simplifying assumption that the collision rate of electrons with heavy ions was substantially greater than with neutral atoms, an assumption which is certainly not valid in the lower ionosphere. However, the effects of collisions with neutral atoms remain unimportant. The losses are increased by such collisions (by a factor of $\sim 10^3$ in the E region) but the corresponding losses due to collisions of *heavy ions* and neutral atoms are much greater, the factor being $\sim 10^5$ (compare σ_0 and σ_3 in Table 1 below).

Equations (2) and (3) are linearized, being valid only when the perturbations $\mathbf{v}$, $\mathbf{v}'$ and $\mathbf{H}_0$–$\mathbf{H}$ are small compared with V and $\mathbf{H}_0$. This condition is satisfied generally, except at several Earth radii where the field may be violently disturbed.

3. The transmission parameters

Solutions of the above equations are sought, of the form $\exp i(\omega t - kx)$. The dispersion equation is (Piddington 1956)

$$\frac{\omega^2}{k^2} = QV^2S^2 + \frac{i}{\epsilon}, \tag{5}$$

where

$$Q = \{1+\eta(1+i\omega\tau)^{-1}\}^{-1} \tag{6}$$

and S has very nearly the values $\cos\theta$ and unity for the ordinary (O) and extraordinary (E) waves respectively (Piddington 1955), θ being the angle between the wave normal and $\mathbf{H}_0$. The term i/ϵ corresponds to absorption due to the finite value of σ_0 (that is, due to electron–ion collisions) and as seen in the Appendix, may generally be neglected. When this is done the dispersion equation becomes

$$k = \frac{\omega}{VS}\{1+\eta(1+i\omega\tau)^{-1}\}^{\frac{1}{2}}. \tag{7}$$

There are three limiting cases, corresponding to different physical situations:

(a) Waves of very low frequency, $\omega\tau \gg 1$. The dispersion equation reduces to

$$k \sim \frac{\omega(1+\eta)^{\frac{1}{2}}}{VS}\left\{1-\frac{i\omega\tau\eta}{2(1+\eta)}\right\} \tag{8}$$

which describe hydromagnetic waves travelling with phase (and group) velocity V_0S, where

$$V_0 = V(1+\eta)^{-\frac{1}{2}}. \tag{9}$$

The absorption coefficient κ (the wave decays as $\exp(-\kappa x)$) is given by

$$\kappa = \frac{\omega^2\tau\eta}{2V_0S(1+\eta)}. \tag{10}$$

The absorption coefficient of a hydromagnetic wave in a *single* gas of conductivity σ is (Piddington 1955)

$$\kappa = \omega^2(8\pi\sigma S^3V_0^3)^{-1}. \tag{11}$$

Putting $S = 1$ and using equation (9), the two expressions for κ are identical, provided

$$\sigma = \frac{(\rho+\rho_n)^2}{\rho_n\tau H^2}, \tag{12}$$

which is the value of σ_3 for the *whole* gas (see Appendix, the σ_0^{-1} term being neglected). Hence the waves described by equations (9) and (10) travel in the composite gas, which may be considered as a fluid of density $(\rho+\rho_n)$ and effective conductivity σ_3 as found in the Appendix.

(b) A second limiting case occurs when $\eta \gg \omega\tau \gg 1$. The dispersion equation then reduces to

$$k \sim \frac{1}{V_0S}\left(\frac{\omega}{2\tau}\right)^{\frac{1}{2}}(1-i), \tag{13}$$

which describes waves travelling with velocity $V_0S(2\omega\tau)^{\frac{1}{2}}$ and suffering absorption

$$\kappa = \frac{1}{V_0S}\left(\frac{\omega}{2\tau}\right)^{\frac{1}{2}}. \tag{14}$$

These waves travel neither in the medium as a whole (their velocity would be V_0S) nor in the plasma alone. Their velocity corresponds to that in a single medium of density $\rho_n/2\omega\tau$, which lies between ρ_n and ρ and varies with the wave frequency so that the medium is dispersive.

Another way of looking at this case is to combine equations (9), (12) and (14) from which, remembering that $\rho_n \gg \rho$, we have

$$\kappa = (2\pi\sigma_3\omega)^{\frac{1}{2}}S^{-1}. \tag{15}$$

Apart from the factor S^{-1}, this is identical with the solution of the equation

$$\nabla^2\mathbf{H} = 4\pi\sigma_3\frac{\partial\mathbf{H}}{\partial t}, \tag{16}$$

for waves of the form required here. This field equation, recognizable from

"skin effect" theory, applies when the conductor does not move. Thus equation (15) corresponds, apart from geometrical complexities, to the situation assumed by Ashour & Price (1948).

(c) A third case of interest is that for which $\omega\tau \gg 1+\eta$, when

$$k \sim \frac{\omega}{VS}\left(1-\frac{i\eta}{2\omega\tau}\right). \tag{17}$$

These waves have velocity VS so that they are travelling in the plasma alone. The absorption coefficient is

$$\kappa = \frac{\eta}{2VS\tau} \tag{18}$$

so that the neutral atom gas is important in determining the absorption rate.

4. A model of the atmosphere and interplanetary medium

In order to permit a quantitative discussion of the results of the previous section a (very incomplete) model of the Earth's atmosphere* is given in Table 1. Included are the number densities N, N_n of the heavy ions and neutral atoms, their average atomic weights A (assumed equal) and the gas temperature T.

At the higher levels h_a, h_b, h_c, whose actual heights are unspecified but may be in the range 3–8 Earth radii, the true conditions are virtually unknown. Level h_a may be that through which "whistling atmospherics" pass and so have an electron (and ion) density $\sim 10^3\,\mathrm{cm}^{-3}$. The temperature is likely to be controlled largely by conductivity downwards and so should be between $\sim 10^3\,^\circ\mathrm{K}$ in the ionosphere and perhaps $\sim 10^5\,^\circ\mathrm{K}$ in interplanetary space. The corresponding proportion of neutral atoms should be ~ 1 in 200 (Nikolsky 1958).

The two columns headed h_b, h_c represent alternative possibilities of conditions at the border between the Earth's atmosphere and true interplanetary space. The corresponding magnetic field strength listed is that at ~ 7 Earth radii under static conditions. The values of $N = 10^3\,\mathrm{cm}^{-3}$ is probably an upper limit and there is evidence that it may be too large (Blackwell 1957). Consequently a second alternative $N = 10^2\,\mathrm{cm}^{-3}$ is also listed. The two corresponding values of N_n are chosen to provide probable upper and lower limits of σ_3. With the maximum value of N is associated the minimum possible value of N_n corresponding to a high temperature in a pure hydrogen atmosphere. The temperature may be as high as $\sim 10^5\,^\circ\mathrm{K}$ (Chapman 1957) in which case a state of equilibrium between ionization by collision and recombination gives $N_n/N \sim 10^{-6}$ (Shklovsky 1951). The ratio is independent of the gas density. The maximum likely value of N_n is associated with the lower value of N (at h_c). The gas temperature may be lower and dust may also play a part in increasing N_n/N to about 1/500 (Shklovsky 1958).

Also listed in Table 1 are the values of magnetic pressure, $H_0^2/8\pi$, and gas pressure at the three upper levels. In each case the magnetic pressure is substantially larger, so that the approximation of neglecting elastic forces in the gas compared with those in the field (neglecting the velocity of sound compared with V) is a reasonable one.

* I wish to thank Dr D. F. Martyn and Dr G. R. Ellis for access to data averaged from available experimental and theoretical sources.

Table 1

A model atmosphere

Altitude	(km)	100	1 000	h_a	h_b	h_c
N	(cm^{-3})	10^5	10^5	10^3	10^3	10^2
N_n	(cm^{-3})	3×10^{12}	2×10^5	5	10^{-3}	0·2
A		30	16	1	1	1
T	(°K)	250	3 000	10^4	10^5	—
τ	(s)	$2{\cdot}0\times10^4$	$1{\cdot}3\times10^4$	$1{\cdot}3\times10^7$	$1{\cdot}3\times10^7$	$\sim1{\cdot}3\times10^8$
H_0	(gauss)	0·33	0·21	0·012	10^{-3}	10^{-3}
σ_0	(e.m.u.)	—	$1{\cdot}6\times10^{-9}$	$1{\cdot}0\times10^{-8}$	$3{\cdot}1\times10^{-7}$	—
σ_3	(e.m.u.)	$6{\cdot}7\times10^{-14}$	$1{\cdot}8\times10^{-20}$	$1{\cdot}7\times10^{-22}$	$1{\cdot}2\times10^{-16}$	$6{\cdot}2\times10^{-22}$
V	(cm s^{-1})	$4{\cdot}2\times10^7$	$3{\cdot}6\times10^7$	$8{\cdot}3\times10^7$	$6{\cdot}9\times10^6$	$2{\cdot}2\times10^7$
V_0	(cm s^{-1})	$7{\cdot}6\times10^3$	$2{\cdot}6\times10^7$	$8{\cdot}3\times10^7$	$6{\cdot}9\times10^6$	$2{\cdot}2\times10^7$
$H_0^2/8\pi$	(dyn cm^{-2})	—	—	$5{\cdot}7\times10^{-6}$	$4{\cdot}0\times10^{-8}$	$4{\cdot}0\times10^{-8}$
NkT	(dyn cm^{-2})	—	—	$1{\cdot}4\times10^{-9}$	$1{\cdot}4\times10^{-8}$	$\gg1{\cdot}4\times10^{-8}$

The table also gives σ_0, the conductivity along the magnetic lines of force; this is the effective transverse conductivity in the absence of all neutral atoms (when $\sigma_0=\sigma_3$, see Appendix).

5. The transmission of geomagnetic disturbances

When the data of Table 1 are substituted in the equations of the previous section the propagation constants of electromagnetic waves through the atmosphere may be determined. This has been done for three wave periods, 1 s, 100 s and 10 000 s, which cover most of the range of observed fluctuations. For each period and for each level of the model atmosphere the propagation velocity ($V(E)$, etc.) is determined and the distance ($\delta(E)$ etc.) in which the wave is attenuated by a factor e^{-1}. The latter variable is the inverse of κ. The results are given in Table 2.

Table 2

Wave velocities and rates of absorption

wave period (s)	1	100	10 000
ω	6·3	$6{\cdot}3\times10^{-2}$	$6{\cdot}3\times10^{-4}$
$V(E)$ cm s^{-1}	$3{\cdot}9\times10^6$	$3{\cdot}9\times10^5$	$3{\cdot}9\times10^4$
V(1 000) cm s^{-1})	VS	VS	$\sim VS$
$V(h_a, h_b, h_c)$	VS	VS	VS
$\delta(E)$ km	6·1	61	610
δ(1 000) km	$4{\cdot}7\times10^6$	$4{\cdot}7\times10^8$	$\sim4{\cdot}7\times10^6$
$\delta(h_a)$ km	$4{\cdot}3\times10^{12}$	$4{\cdot}3\times10^{12}$	$4{\cdot}3\times10^{12}$
$\delta(h_b)$ km	$1{\cdot}8\times10^{16}$	$1{\cdot}8\times10^{16}$	$1{\cdot}8\times10^{16}$
$\delta(h_c)$ km	$2{\cdot}9\times10^{13}$	$2{\cdot}9\times10^{13}$	$2{\cdot}9\times10^{13}$

An examination of the above two tables shows that from the point of view of the transmission of geomagnetic disturbances, the atmosphere may be divided into two parts, separated by a transition region.

(i) At the 100 km level the value of $\omega\tau$ does not exceed $\sim 10^5$ while η has a value 3×10^7, so that for waves with periods in the range listed transmission is

always by mode (b) of Section 3. In this mode the gas as a whole does not move appreciably but the magnetic field diffuses through the gas. A disturbance suffers a time delay in passage and, if oscillatory in form, its amplitude is reduced.

This lower (L) or diffusion region extends from the ground up to levels of a few hundred kilometres depending on the value of ω. The transition region occurs where $\omega\tau \sim \eta$. For waves of period 1 s this is between 200 and 300 km; for waves of period 10^4 s it is between 500 and 1 000 km. The L region medium is dispersive.

(ii) Above the transition region we have $\omega\tau \gg 1+\eta$ and transmission is by mode (c) of Section 3, that is by hydromagnetic waves travelling in the plasma alone, the latter moving freely through the neutral atom gas. This upper (U) region extends into interplanetary space. Absorption effects in this region are generally negligible.

Although absorption in the U region is generally unimportant, transmission may be very complex. In the first place, even in a uniform medium, the O hydromagnetic wave* is transmitted anisotropically. The Poynting flux, averaged over a cycle, flows along the magnetic field lines. Thus a disturbance having a configuration corresponding to the O wave and originating several Earth radii away would tend to be conducted exclusively into regions of high latitude. This may explain the greater amplitudes of some disturbances in polar regions and is discussed in connection with micropulsations in Section 9.

On the other hand the E wave travels freely in any direction. The collision of a solar gas cloud with those lines of force lying near the noon meridian will initiate an E wave which travels to all parts of the Earth, increasing the horizontal component of the field as observed. The delay in reaching the dark side of the U region corresponds to a velocity of propagation $10^7 - 10^8\,\mathrm{cm\,s^{-1}}$ and a distance $\sim 2\times10^4$ km giving a time of 20 – 200 s.

In addition to the peculiarities of the O and E wave in a uniform medium, one must also take into account the fact that the values of ρ, ρ_n and $\mathbf{H}_0$ all vary, often changing substantially within a fraction of a wavelength. There should therefore be refraction and reflection effects including standing waves. An examination of these phenomena may be made by the methods of Ferraro (1954) and Ferraro & Plumpton (1958).

These conclusions suggest that the analyses of Ashour & Price (1948), Sugiura (1950) and others might require considerable extension before they were applicable to disturbances originating above a few hundred kilometres.

6. The movements and locations of the geomagnetic current systems

Before proceeding further it is useful to prepare a simplified model of the L and U atmospheric regions in the equatorial plane. This is shown schematically in Figure 1a, the magnetic lines of force being shown by the dashed lines. The limits of the U region are denoted by the planes A and B which may move but must at all times contain fixed quantities of gas and magnetic flux. The limits of the L region gas are C and G, which may not move at all under the influence of disturbance fields. However, the field normally contained between C and G may be increased or decreased by a downward or upward movement of the U region, which may penetrate into, or separate from, the L region.

* The O wave has a perturbation magnetic field ($\mathbf{H}-\mathbf{H}_2$) perpendicular to the plane containing the wave normal and the steady field $\mathbf{H}_0$; the E wave has its field in this plane.

Suppose now that equilibrium conditions are disturbed by the application of a force F on the upper boundary of the U region as in Figure 1b. This might be due to the arrival of a solar gas cloud as in the Chapman & Ferraro (1931) theory of the first phase of a sudden commencement (*SC*) storm. An electric current flows in the region A corresponding to the initial compression of the field and given quantitatively by

$$\text{curl } \mathbf{H} = 4\pi\mathbf{j}. \tag{19}$$

This current does not directly cause a geomagnetic disturbance because of the effectiveness of the U region as a shield. However, it does result in a force of density $\mathbf{j} \wedge \mathbf{H}$ on the gas and so causes a downward travelling hydromagnetic E wave which has its own current system. When the wave reaches level B it causes

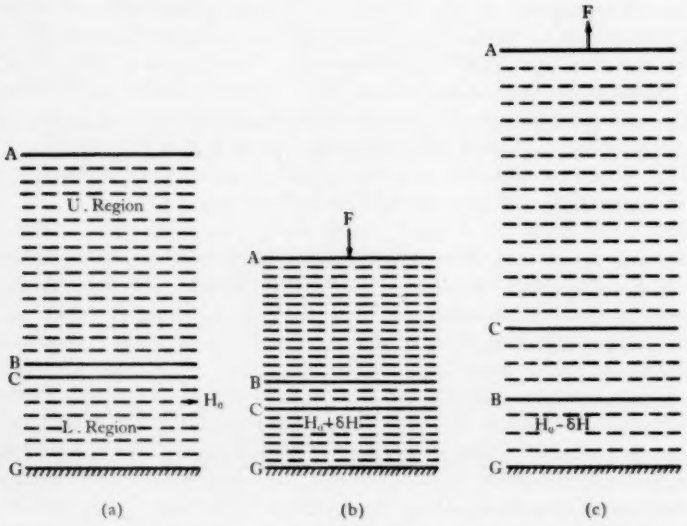

FIG. 1.—The upper and lower atmsopheric regions. The Earth's magnetic field (shown as dashed lines) is frozen into the U region but moves through the L region.

the gas and field of the U region to interpenetrate the *gas* of L region. The *field* of L region is compressed and so the disturbance field diffuses through to the ground level G.

It is the current system in the L region which is directly responsible for the observed geomagnetic disturbances. The Earth is effectively shielded from any current systems located in the U region. This result holds equally for regions away from the geomagnetic equator; the geometry of the field is more complicated but the general conclusion is unchanged.

It has often been suggested "that important sources of the field of geomagnetic storms were located in the atmosphere" (Forbush & Vestine 1955; by atmosphere they mean the region below $\sim$ 1 000 km). The present analysis shows that the currents directly responsible for *all* geomagnetic disturbances must flow below $\sim$ 1 000 km. This result might provide an argument against the atmospheric dynamo theory of magnetic disturbances developed by Wulf and Vestine (see

Vestine 1953). Observational evidence that some currents flowed in the ionosphere favoured a purely ionospheric origin of these currents. It would seem, however, that this result is explained equally well wherever the original source of the disturbance.

The above simple picture of a current system propagating through the U region by hydromagnetic action is not, alone, sufficient to account for all observational data. For example, Forbush & Vestine (1955) have shown that the increase of the horizontal component during an SC increase is anomalously large at the equator during the day and larger on those days when the diurnal variation S_q is high. This is probably due to the high conductivity of the ionosphere at the times and place referred to, together with an electric field system associated with the high-latitude disturbance. An electric field system at levels below a few thousand kilometres might explain other inferred current systems and also the aurora.

An electric field system might be introduced near the poles, either by an influx of positive and negative ions along different paths or in association with a wave, presumably hydromagnetic. The former method meets certain difficulties which are discussed in the following section. An apparent objection to the latter lies in the fact that a plane hydromagnetic wave in the U region has a very weak electric field. This is because of the very high "effective" conductivity σ_3 in the plasma alone, allowing a weak electric field $\mathbf{E}$ to drive the current required. However, it is well known that the effective conductivity is σ_3 only when the Hall current $\sigma_2 H \times E/H$ is prevented from flowing by charge accumulations on boundaries. When this current may flow in a closed circuit, then the effective conductivity may be reduced. Waves with this characteristic may possibly have associated electric fields large enough to account for aurorae and the current systems of polar origin. Such waves comprise a twisting motion of a bundle of magnetic lines of force and would seem to warrant further study as a possible factor in magnetic disturbance theory.

7. The penetration of the solar cloud into the Earth's magnetic field and the location of the auroral zones

In the previous sections the transmission of geomagnetic disturbances from several Earth radii has been briefly considered. It is likely that most of these disturbances originate in the region of collision between the solar gas cloud and the Earth's magnetic field and a study of this collision region is an essential part of the transmission problem. In particular, we are concerned with the degree to which the solar gas cloud *penetrates* the Earth's field in addition to *distorting* it. As will be evident later, the question of penetration is the central problem in the theory of the main phase of a storm and is highly significant in auroral theory. In order to avoid confusing "penetration" with "distortion" the former is defined here by the co-latitude θ of the point on the Earth's surface and in the meridian plane passing through the Sun, cut by the magnetic line of force which just touches the surface of the cloud. Lines of force lying in the same plane and cutting the Earth at points with smaller co-latitudes will all penetrate the cloud; those with larger co-latitudes will not.

The problem of the approach of a cloud with a plane surface towards a dipole has been considered by Chapman & Ferraro (1931). Near the Sun–Earth line the dipole field is compressed and the forward motion of the gas is arrested.

North and south of this line the opposing force falls to zero before rising again so that "horns" of gas develop on the surface of the cloud, penetrating freely into regions of high latitude (see Chapman & Ferraro's Figure 4).

When this theory was developed it was not known that the Earth's field was permeated by a highly conducting plasma. The presence of the plasma necessitates important changes; in particular it eliminates the horns of gas. When a rigid, perfectly conducting sheet approaches a dipole in free space the magnetic field ahead of the sheet is found by adding the fields of the original dipole and an image dipole behind the sheet. Components of the two fields pointing in opposite directions cancel one another. This means that at the points on the sheet where the original field is perpendicular to the sheet the field becomes zero and if the sheet were replaced by the surface of a gas cloud, horns should develop around these points. This means, in effect, that as a particular line of force of the original field becomes perpendicular to the sheet it is "cut" and instead of remaining a loop its ends go to infinity in the North and South directions as sketched by Chapman & Ferraro.

In the real problem the Earth's field is "frozen into" a plasma and the field due to currents in the cloud is "frozen out of" that plasma. If the plasma were perfectly conducting the separation would be complete and none of the lines of force of the Earth's field would be cut. Even those lines leaving the Earth at the magnetic poles would remain complete loops, suffering distortion but excluding the solar gas. The presence of a magnetic field in the cloud before it nears the Earth would not affect this conclusion. Assume that initially the face of the cloud is a plane; ahead of the cloud a hydromagnetic compression wave is formed and traverses the U region as the cloud is brought to rest. In the quasi-stable situation ensuing, the whole of U region is compressed but is not invaded by any external field or gas.

Near the Sun–Earth line the forward motion ceases when the hydromagnetic pressure difference across the face of the cloud equals the momentum destroyed per second. If we can neglect gas pressure gradients then

$$H_e^2 = 8\pi\rho u^2 \tag{20}$$

where H_e is the strength of the distorted field just ahead of the cloud and ρ and u are the mass density and velocity of the solar plasma. This result follows whether or not the dipole field is frozen into a plasma. However, the physical situations are different in the two cases: in the former case the plasma and field are *compressed* while in the latter case the field is increased by the *invasion* of a field of external origin which is parallel with the dipole field.

Away from the Sun–Earth line the different physical situations are really important because the oppositely directed fields, which would cancel one another in a vacuum, are kept apart by the plasma. Thus, cutting of the lines of force and penetration by the solar gas is prevented.

The forward motion of the neutral atoms in the solar cloud is not arrested by the Earth's field. They continue on into the atmosphere where some are ionized (they will not lose much energy in this process) and trapped in the field. These protons have energy up to a few times 10^4 eV.

Since the solar plasma alone, neglecting neutral atoms, has finite conductivity (σ_3), the compressed magnetic field will diffuse through its surface. The situation, neglecting distortion of the surface of the cloud away from the Sun–

Earth line, is shown in Figure 2. The surface is at rest in the plane $x = x_0$ with a particular magnetic line of force L_θ touching its surface and cutting the Earth's surface in co-latitude θ. The direction of the field near $x = x_0$ is parallel to Oz. The equation giving the rate of diffusion of the field into the cloud is equation (16) above. If diffusion starts at time $t = 0$, then the field in the plasma is given, after integration, by

$$H(x) = H_c \exp(-\pi\sigma_3(x-x_0)^2/t). \tag{21}$$

The total magnetic flux through the cloud and lying between parallel planes $y =$ const., separated by 1 cm, at time t is

$$\psi = \int_{x=x_0}^{\infty} H(x)dx.$$

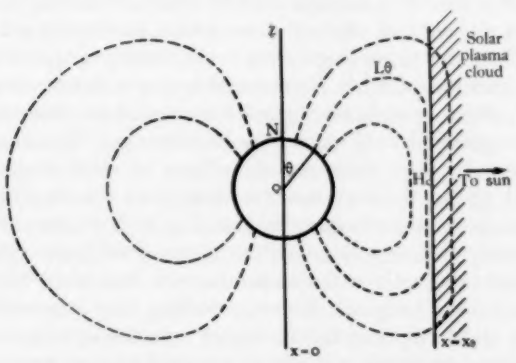

FIG. 2.—Distortion of the lines of force of the Earth's field by a solar gas cloud.

Using equation (21) and integrating

$$\psi = H_c\left(\frac{t}{4\sigma_3}\right)^{\frac{1}{2}}. \tag{22}$$

Through the geometry of the *undistorted* dipole field of the Earth, the flux ψ may now be related to the co-latitude θ of the line of force L_θ. The first relevant equation is

$$\psi = M\int_{x_\theta}^{\infty} x_\theta^{-3}dx \tag{23}$$

where M is the Earth's dipole moment and ψ the flux per cm thick sheet lying beyond a distance x_θ which, in turn, is related to the co-latitude by

$$x_\theta = a\sin^{-2}\theta \tag{24}$$

where a is the Earth's radius. Combining these equations and putting $M = 8{\cdot}2 \times 10^{25}$ and $a = 6{\cdot}37 \times 10^8$, we have

$$\sin^8\theta = 2{\cdot}5 \times 10^{-17} H_c^2 t\sigma_3^{-1}. \tag{25}$$

The value of H_c is found from equation (20); for a stream of (large) proton density 1 000 cm^{-3} and velocity 10^8 cm s^{-1} it is 0·02 gauss. The value of σ_3 is at least 10^{-8} e.m.u. and if $\theta = 23°$, corresponding to the auroral zone, the time taken to penetrate is $5{\cdot}5 \times 10^8$ s or 18 years. The time available for penetration is a few hours so that according to the above analysis the flux which may penetrate the cloud is much less than that between the pole and the auroral zone; it corresponds to that down to co-latitude of about 6°.

This result raises some doubt concerning Martyn's (1951) theory of the location of the auroral zone. Martyn's criterion of penetration was that the stream advanced into the Earth's field until the pressure exerted by the lines of magnetic force equalled the pressure of the stream. In the above analysis this is a criterion of *distortion* rather than *penetration* of the field.

Although the above discussion applies quantitatively only near the Sun–Earth line it seems applicable to theories of the direct entry of gas into auroral regions. Whatever path is followed by this gas it must cross a total magnetic flux given by equations (23) and (24) and when θ corresponds to the auroral zones ($\sim 23°$) this amount is about 10^4 times larger than is allowed by simple diffusion theory.

There are a number of other factors which may influence the penetration of the solar cloud; these have been neglected in the above discussion and in the earlier treatments referred to (Chapman & Ferraro 1931, Chapman 1952, Ferraro 1952, Martyn 1951). First there are the effects of electric space-charge fields which will build up on the east–west boundaries of the Earth's magnetic field and also perhaps in small-scale distributions due to the inherent instability of a medium comprising two or more separate electron streams. There is also the possibility of hydromagnetic turbulence near the boundary between the fast-moving cloud and the U region. Beyond recalling that large-scale space-charge fields tend to be short circuited by the highly conducting surrounding gas (Piddington 1958), these questions will not be pursued further here. The results of the next section suggest that some mechanism allows substantial penetration, but only after a period of about half a day.

8. The main phase of an *SC* storm

The main phase of an *SC* storm is a worldwide decrease in magnetic intensity lasting for a day or two. If the space outside the Earth's surface were a vacuum then the decrease would be similar to that caused by an electric current flowing near the geomagnetic equator in a ring of radius a few Earth radii. Chapman & Ferraro and later Martyn (1951) have invoked such a ring current to explain the main phase and other storm phenomena.

However, the space outside the Earth is not a vacuum and the presence of ionized gas greatly modifies the problem, as pointed out by Parker (1956). Parker and also Singer (1956) criticized the ring current theory on the grounds that the conductivity of the interplanetary gas is so high that the field due to the ring current would take months or years to penetrate to the Earth. Strictly speaking this objection is invalid because the effective electrical conductivity of the gas concerned is very low (Appendix and Table 1). However, because of the large collision period τ, the fully ionized plasma must, in this particular problem, be considered to move independently of the neutral atoms. The plasma alone has a high conductivity and is impervious to magnetic fields in the period of time available.

As seen above, the presence of this plasma necessitates changes in the theory of the *SC*; corresponding changes are required in the theory of the main phase. The immediate effect of a ring current is to reduce the original field just inside the ring and increase the field just outside. This localized effect is transmitted downwards to the Earth hydromagnetically, the whole of *U* region being lifted and the field in *L* region then reduced. The situation is illustrated in Figure 1(c) where the *U* region has been lifted to leave a space *BC* between the *U* and *L* regions. Such a space is only formed in the simplified model; in nature there is no surface of discontinuous change in properties of the medium; one merges into the other.

The fundamental physical feature of the main phase is an outward radial mechanical force which raises the *U* region. Such a force is not provided in theories where the ions and electrons spiral around the Earth in opposite directions (Singer 1956). The ions and electrons must move in the *same* direction with nearly the same velocity so that for a given current the centrifugal force is sufficient to balance the Lorentz force. This condition is satisfied in Martyn's (1951) discussion of the ring current theory. There appears to be a difficulty, however, caused by the presence of an electrostatic field which must exist as a necessary part of the mechanism (to compensate for the different centrifugal forces on ions and electrons). This field is experienced by the gas occupying the ring volume prior to the entry of the solar ions whose movement provides the ring current. The effect of the crossed electric and magnetic fields, E and H, on the plasma is to accelerate it to velocity E/H; this velocity turns out to be the same as that of the circling solar ions (Martyn 1951). The main phase of a storm appears to build up gradually from zero so that initially $E = 0$ and the velocity of drift is zero. This would mean that individual solar ions entering the ring must share their kinetic energy with all other ions in the ring to provide a gradual increase in E and a corresponding acceleration of the gas. It is difficult to see how this could occur, as it would mean that the ions were giving up their energy to a steady magnetic field in the absence of an electric field.

Any explanation of the main phase must depend primarily on *penetration* of the solar gas into the geomagnetic field. Only after penetration can the solar gas exert an outward pull on the field. There appears to be a simpler explanation of this pull than the delicately balanced ring current: the magnetic lines of force which diffuse into the solar gas may be carried away by the gas flowing past the Earth to provide a field something like that of Figure 3. Arguments in favour of the formation of such a field are similar in some ways to those given for a radial solar field (Piddington 1958). Once a magnetic tube of force enters the solar gas which is sweeping past the Earth it is frozen into that gas and drawn out beyond the Earth's orbit. The additional magnetic energy so created is provided at the expense of the kinetic energy of the cloud. A trailing field of strength H_t exerts a pull of $H_t^2/8\pi$ per cm^2 on the receding cloud.

On the above hypothesis the sequence of events during an *SC* storm (ignoring special polar effects) is as follows. The solar cloud envelops and sweeps past the Earth's field compressing it on the solar side. Near the geomagnetic equator the field is everywhere nearly parallel with the surface of the cloud as in Figure 2 and the force $F = H_c^2/8\pi$ is transferred downwards to the surface of the Earth as in a purely hydrostatic problem. At the surface of the Earth it increases the magnetic pressure by approximately the same amount at all longitudes. As time passes, the solar gas penetrates the field so that its ordered motion of 10^8 cm s^{-1}

is converted, at least in part, to disordered motion. The corresponding temperature of the protons is 4×10^7 °K, so that a small amount of gas can expand the field until its outermost lines of force can be blown away. The lines of force so drawn out will cut the Earth at all longitudes because a reduction of magnetic pressure within a particular range of longitudes will cause an influx of field from outside that segment. The result is a decrease in magnetic pressure, and field strength, at all longitudes which, when it grows to exceed the increase due to the force F, constitutes the main phase of the storm.

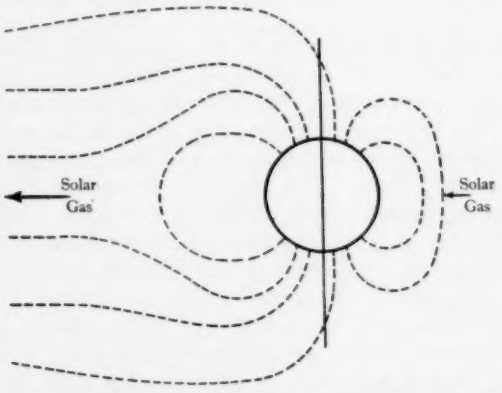

FIG. 3.—Distortion of the lines of force of the Earth's field during the main phase of an SC storm.

Clearly the penetration is the central problem and its mechanism is not evident. Nevertheless it is possible to estimate, very approximately, the degree of penetration. Let H and h be the undisturbed field and the decrease during the main phase. The decrease in magnetic pressure is then everywhere $\frac{1}{8}\pi\{H^2-(H-h)^2\}$. At the top of the U region where the field is entirely torn away we have $h = H = H_d$ say, the pressure change being $H_d^2/8\pi$. If this is equated to the change at ground level, where $h \ll H_d$ then we have approximately

$$H_d^2 = 2hH = 0{\cdot}66h.$$

For a storm of 100γ ($h = 10^{-3}$ gauss), $H_d = 0{\cdot}026$ gauss, which corresponds to the undisturbed dipole field at a distance of $z \sim 2{\cdot}4$ Earth radii. The lines of force cutting the geomagnetic equatorial plane at this distance return to the Earth at co-latitude given by $\sin\theta = (2{\cdot}4)^{-\frac{1}{2}}$, $\theta = 40°$.

These results, which may be only rough approximations, suggest two things. The value of H_d is much larger and the value of z much smaller than found from ring current theory. This is because the geomagnetic disturbance is transmitted through the U and L regions as a hydromagnetic pressure change rather than directly as an invading magnetic field of external origin. Thus the disturbance at a few Earth radii may be much more violent than previously believed. The second item of significance is the large value of θ, suggesting a mechanism of penetration much more efficient than that dealt with above.

These ideas will be discussed in more detail elsewhere, together with possible significance in connection with Gegenschein, whistling atmospherics and cosmic ray variations.

9. Micropulsations

If the ordinary equation for the rate of absorption of hydromagnetic waves in a single fluid (equation (11) above) is applied for waves of periods 100 s or less in the *E*-region, they are damped out in a metre or less. This result seemed to provide an objection to the otherwise attractive suggestion (Dungey 1954, Lehnert 1956, Kato & Akasofu 1956, Obayashi & Jacobs 1958 and others) that micropulsations resulted from hydromagnetic waves travelling to the Earth from interplanetary space. However, according to the above theory, electromagnetic waves of the periods concerned do not traverse the *E*-region as hydromagnetic waves but rather diffuse through a stationary conductor without excessive losses.

This result, if correct, would invalidate the theory of Dessler (1958) who assumed that the disturbances travelled as hydromagnetic waves in the ion-electron plasma alone. He found a minimum wave velocity at ~ 400 km of $\sim 10^7$ cm s^{-1} followed by a rapid increase in velocity at lower levels (due to the decreasing ion density). According to Table 2, however, the velocity is lower at lower levels because of the effects of neutral atoms.

As mentioned in Section 5, some properties of micropulsations may be explained in terms of *O* type hydromagnetic waves. Referring to Figure 2, the deformation of the Earth's field at the plane $x = x_0$ may propagate directly to the Earth as an *E* wave. Now consider lines of force lying in the plane $x = 0$. The gas cloud moves past the outer edges of these lines and must deform them so that they lie on a curved sheet whose inner portion remains in the $x = 0$ plane and whose outer portion is nearly parallel with the $y = 0$ plane. The main deformation occurs near the region where magnetic and inertial forces balance according to equation (20)*. The deformation field is parallel to Ox so that a hydromagnetic disturbance may only travel to the Earth along the magnetic lines of force. As Martyn (1951) has shown, reasonable values of ρ and u in equation (20) give H_c corresponding to a line of force leading into the auroral zone. This might explain the fact that micropulsations are observed most frequently in those latitudes.

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APPENDIX

The effective electrical conductivity of the outer atmosphere and interplanetary space

An ionized gas permeated by a magnetic field $\mathbf{H}_0$ is anisotropically conducting, having conductivity components σ_0 (parallel to $\mathbf{H}_0$), σ_1 (Pedersen) and σ_2 (Hall). The "effective" conductivity, which determines the rate of diffusion of a magnetic

* The physical concept behind this equation differs from that given by Martyn and by Obayashi & Jacobs (1958) who compare gas and field energy densities. It is similar to that of Warwick (1957), when his gas slab has thickness (h) equal to the radius of curvature (r) of the plane containing the line of magnetic force.

field through the gas, is then $\sigma_3 \equiv \sigma_1 + \sigma_2^2/\sigma_1$ (Piddington 1955, 1956; Cowling 1956).

In the particular case of a fully ionized gas $\sigma_3 = \sigma_0$ and the effective conductivity for hydrogen is given approximately by

$$\sigma_0 \sim 10^{-14} T^{3/2} \text{ e.m.u.} \tag{26}$$

where T is the kinetic temperature. At a distance of two or three Earth radii where the temperature may be about 10^4 °K the conductivity would then be about 10^{-8} e.m.u. The time taken for a magnetic disturbance to penetrate to the Earth by a process of diffusion through a stationary medium would be some years.

However, the gas concerned cannot be *fully* ionized; there must be a balance between the processes of ionization and recombination, with some neutral atoms present. It will be seen that a very small proportion of such atoms, say 1 in 10^{10}, may reduce the effective conductivity below that given by equation (26).

Neglecting the unimportant dissipative effect of electron-neutral atom collisions, a general expression for the effective conductivity of a gas is

$$\frac{1}{\sigma_3} = \frac{1}{\sigma_0} + \frac{\rho_n \tau H^2}{(\rho + \rho_n)^2}, \tag{27}$$

where ρ, ρ_n are the mass densities of the ion plasma and neutral atom gas and τ is a collision period given by

$$\tau = \frac{3}{8\rho r^2} \left(\frac{m_1 m_2 (m_1 + m_2)}{2\pi k T} \right)^{\frac{1}{2}}. \tag{28}$$

Here r is the collision distance for heavy ions and neutral atoms and m_1, m_2 the masses of those particles. The physical significance of the unusual collision period chosen is discussed below (see also Piddington 1957, Appendix I).

Accurate values of r for the conditions concerned are not available but a reasonable estimate is probably given by (Cowling 1945):

$$r = 10^{-8} K \left(\frac{300}{T} \right)^{\frac{1}{2}}, \tag{29}$$

where $K \sim 6$ in the Earth's true atmosphere and $K \sim 1$ at a few Earth radii where the gas is presumably mainly hydrogen. Combining equations (26) to (29) we have

$$\frac{1}{\sigma_3} = \frac{10^{14}}{T^{3/2}} + \frac{8{\cdot}1 \times 10^{33} N_n H^2}{N(N+N_n)^2 A^{\frac{1}{2}} K^2} \tag{30}$$

where N, N_n are the number densities of ions and neutral atoms and A their atomic weight, assumed equal.

The values of the various quantities determining σ_3 are discussed in Section 4. For the present purposes a set of likely values, corresponding to a distance of a few Earth radii are chosen as follows: $N = 10^3 \text{ cm}^{-3}$, $N_n = 10^{-3} \text{ cm}^{-3}$, $A = K = 1$; neglecting the small term in σ_0, equation (30) then reduces to

$$\sigma_3 = 1{\cdot}2 \times 10^{-22} H^{-2}. \tag{31}$$

The value of H in the equatorial plane is about $0{\cdot}33 z^{-3}$ where z is the distance in units of Earth radii. At say 5 Earth radii we have $\sigma_3 = 1{\cdot}7 \times 10^{-17}$ e.m.u.; the

value of H is of the order of observed geomagnetic fluctuations. The value of σ_3 decreases as the Earth is approached until the ionospheric E region is reached, where it rises to a maximum value of about 10^{-13} e.m.u. (Baker & Martyn 1953).

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Travel-times of *PKP* up to 145°

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(Received 1959 March 16)

Summary

Examination is made of some 900 *PKP* readings in the range $110° < \Delta < 145°$, as recorded in the *International Seismological Summary* from 24 southern earthquakes. This provides new evidence that the J.-B. times for the *DE* branch before 140° need an increase of about a second. There is no evidence in this range of significant differences between *PKP* times to Europe and North America.

In the range $135° < \Delta < 145°$, a concentration of *PKP* observations preceding the *DE* branch is found fitting the form

$$t = 19\,\text{min}\,29{\cdot}1s + 2{\cdot}69(\Delta - 141°{\cdot}6).$$

This possibly corresponds to the *BC* branch of *PKP* and its extension for the diffracted waves associated with the caustic at *B*.

1. Introduction

One significant difference between the independent solutions of Jeffreys (1939a) and Gutenberg (1951) for the variation of *P* seismic velocity with depth in the Earth occurs near 5 000 km. Jeffreys's solution has a negative *P* velocity gradient between depths of 4 982 km and 5 121 km. At 5 121 km there is a discontinuous velocity increase marking the boundary of the inner core. Gutenberg however finds that no velocity solution consistent with his 1951 travel-time data gives "a decrease in velocity or a strictly vertical segment".

At depths near 5 000 km both solutions depend considerably upon observed travel-times of *PKP* phases, particularly the phase *PKIKP* which enters the inner core and is of *P* type throughout its path. Recently, Bullen & Burke-Gaffney (1958) have shown that the 1954 hydrogen bomb explosions yielded some important additional data on *PKP* travel-times. The purpose of the present work is to present further information, obtained with satisfactory precision from *PKP* observations from natural earthquakes.

The Jeffreys-Bullen *PKIKP* travel-times depend largely on *PKP* readings from the Solomon Islands earthquake of 1932 January 9 (Jeffreys 1939b). They were subsequently checked from observations of the Celebes Sea earthquake of 1934 June 29 (Jeffreys 1942). The present paper presents an analysis of nearly 900 *PKP* observations over the range $110° < \Delta < 145°$ from 24 selected southern earthquakes, and provides some support for the results of Jeffreys.

2. Theoretical *PKP* travel-time curve

The form of the *PKP* travel-time curve is shown in Figure 1. It is divided into four branches by the three cusps *B*, *C* and *D*.

A ray which just grazes the outer core is refracted to *A* at $\Delta \approx 180°$; steeper rays give in turn the branches *AB* and *BC* until at *C* the ray is tangent to the inner core. The cusp *B* at $\Delta \approx 142°$ is similar to an optical caustic. The receding branch *CD* corresponds to rays reflected by the inner core, and *DEF* to the *PKIKP* rays refracted through the inner core.

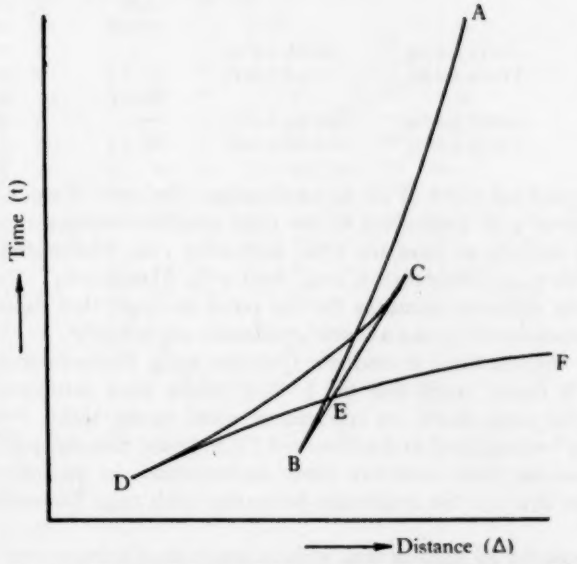

FIG. 1.—Theoretical travel-time curve for the core phase *PKP* (Jeffreys & Bullen 1948, Figure 2).

3. Reduction of readings

Three criteria were used for selection of earthquakes:

(a) occurrence in the Celebes to New Hebrides region over the period 1937 to 1941;

(b) at least 12 readings of *PKP* for $\Delta < 145°$ given in the *International Seismological Summary*; and

(c) no evidence of multiplicity.

Only the 24 earthquakes listed in Table 1 were satisfactory. Six others were rejected as multiple after examination of the Riverview records and the residuals at all stations.

Table 1

The selected earthquakes

1937 January 23	1938 May 30	1939 October 17
January 25	June 9	1940 February 20
July 2	August 31	March 21
August 5	October 20	May 28
September 15	1939 January 30	September 26
September 27	February 3	1941 September 4
1938 February 1	April 5	September 9
May 12	August 18	November 8

The *I.S.S.* solutions were altered in six cases by the method of Jeffreys (1939c, p. 131) and retained in eighteen. If x and y are the north and east displacements to the *I.S.S.* epicentre, the altered solutions are:

	$x°$	$y°$	Revised focal-depth km	Revised time of origin h	min	s
1937 Sept. 15	—	—	78±8	12	27	35·1±0·7
1939 Feb. 3	−0·31±0·03	+0·06±0·05	—	5	26	19·9±0·4
1940 Sept. 26	+0·14±0·01	−0·41±0·19	150±3	3	56	33·8±0·4
1941 Sept. 4	—	—	88±5	10	21	46·0±0·6
1941 Sept. 9	−0·58±0·02	−0·14±0·24	—	7	19	47·2±0·5
1941 Nov. 8	+0·53±0·03	+0·03±0·10	86±3	23	37	31·7±0·6

Using the adopted solutions of all 24 earthquakes, the ratio of the number of *P* residuals between ±4 s (inclusive) to the total number recorded at each of the following near stations is: Brisbane 0·82, Riverview 1·00, Melbourne 0·68, Adelaide 0·70, Perth 0·43, Christchurch 0·92, Apia 0·83, Manila 0·85. These stations have sufficiently different azimuths for the ratios to imply that significant *E–W* or *N–S* systematic errors in the adopted epicentres are unlikely.

For Pacific sources there is evidence (Jeffreys 1954, Burke-Gaffney & Bullen 1958, Carder & Bailey 1958) that the J.–B. *P* tables need corrections of about −2 s. Thus, for each shock, an adjustment equal to the mean *P* residual for $80° < \Delta < 105°$ was applied to the observed *PKP* times; this also partly removed discrepancies arising from incorrect depth assessments. In general, the adjustments were less than 1 s, the maximum (occurring with 1941 November 8) being 2·1±0·7 s.

PKP residuals for all stations with vertical component instruments were calculated using the J.–B. tables for the branch *DEF* (adjusted for ellipticity). All phases within 20 s of these J.–B. times were included. The summaries of the frequencies of the residuals for European and North American stations are shown in Tables 2 and 3. (For statistical convenience the length of the range interval has been twice reduced.)

Tables 2 and 3 exhibit the following prominent features.

(a) For $110° < \Delta < 140°$, there is a main concentration of observations, taken to refer to the *DE* branch of *PKP*, with its mode between 1 s and 2 s. As found by Miss Lehmann in her study (1958) of the Celebes shock of 1948 March 1, there was no concentration for $\Delta < 110°$.

(b) For $135° < \Delta < 145°$, there is a separate distribution of residuals from observations which clearly occur before the main concentration noted in (a). These are possibly associated with the branch *BC* of *PKP* and will be treated in Section 5.

4. Travel-times for the *DE* branch (*PKIKP*)

Before combining the summaries in Tables 2 and 3, a test was made to determine whether *PKIKP* travel-times to the same distance along European and North American paths are equal. The means in Tables 2 and 3 were formed after allotting unit weight to each observation in the square brackets. (Detailed analysis indicates that this gives satisfactory estimates of the mean and its standard error in each range.) The most significant difference between the means is −1·0±0·79 s for $132°{\cdot}6 < \Delta < 135°{\cdot}0$, and for the 8 differences, $\chi^2 = 8{\cdot}8$. Allowing for

Table 2

Summary of frequencies of residuals (in seconds) from PKP observations at European stations

Range (Δ°)	< −14	−14	−13	−12	−11	−10	−9	−8	−7	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6	7	8	9	> 9	Mean residual (s)
110·1–115·0	4		1							1		1		[1	2	2	3	2	4	1]		1	2	1		1	+1·3 ± 0·45
115·1–120·0			1							2	1	1		[2	3	3	1	2	4	4	1]	1	1	1	1		+1·6 ± 0·50
120·1–125·0									2	1			[1	1	4	8	11	8	6	2	6	3	2	1]		3	+2·1 ± 0·33
125·1–130·0						1					2		[3	3	3	11	11	14	7	4	2	3]			2	1	+1·4 ± 0·27
130·1–132·5					1		1			1		1		[2	3	5	3	7	6	3	1]		1				+1·5 ± 0·34
132·6–135·0				1				1				[1	2	1		5	9	3	4	2	1	1]					+1·3 ± 0·42
135·1–137·5							{2	1}					[1	2	1	4	4	6	1	3	4	4	1]		1		+2·4 ± 0·48
137·6–140·0			{1				1	4	5		1	1}	[2	1	2	1	9	4	4	3	2]	1	1				+1·5 ± 0·40
140·1–141·0	1					{1	1		4		3	2		1}	2	1		2		1		1			1	1	
141·1–142·0	1							{1		2	4	2	4	1	1		2	1}			1					2	
142·1–143·0			1						{1	1	3	3	3	4		1	1	1}		1							
143·1–144·0	1								{1	1		3	3	3	3	2	1	1}			2				1		
144·1–145·0											{1		2	5	2	2	1	1		1}			1				

Table 3

Summary of frequencies of residuals (in seconds) from PKP observations at North American stations

Range (Δ°)	<−14	−14	−13	−12	−11	−10	−9	−8	−7	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6	7	8	9	>9	Mean residual (s)
110·1–115·0	1						1								[1	1	1	2	2]					2			+1·4±0·53
115·1–120·0													[1		1	3	2	6	4		4	2]					+2·3±0·47
120·1–125·0													[2	2	3	14	8	2	5	7	1	1]		1	1		+1·2±0·31
125·1–130·0													[3		5	8	10	6	1	2	3	1]				1	+1·0±0·34
130·1–132·5				1										[1		3	6	4	2	1	2]			1			+1·7±0·39
132·6–135·0		{1	1		1	1}							[1	2	3	1	3	3	1	2	2	1	4]	1			+2·3±0·66
135·1–137·5	{1								1		2}			[1		2			3	1	1]		1		1	+2·0±0·79
137·6–140·0		{1		2	1	1		2	2}						[1	1	4		1	2	2]						+2·2±0·60
140·1–141·0								{1}							1	1				1							
141·1–142·0												{1}															
142·1–143·0														{1	1		1}										
143·1–144·0															{1}												
144·1–145·0											{1				3}												

uncertainty in the epicentres it follows that *PKIKP* travel-times for the two paths are equal within about 1 s.

The two sets of *PKIKP* residuals in square brackets may now be combined. Following the statistical method of Jeffreys (1939c, p. 162), the law of error for the residuals x_i is taken to be

$$f(x_i) = \pi^{-\frac{1}{2}}(1-m)h \exp\{-h^2(x_i-\alpha)^2\}+mg(x_i),$$

where h is the precision constant, α the weighted mean, and $g(x_i)$ a slowly varying function estimated using the method of uniform reduction. Taking six ranges at intervals of 5°, assumed means of 1,2,1,1,1,2 s respectively were subtracted and the frequencies then totalled to give the summary in Table 4.

Table 4

Observed and calculated frequencies of the combined PKIKP residuals for $110°{\cdot}1 \leqslant \Delta \leqslant 140°{\cdot}0$

x_i (sec)	−10	−9	−8	−7	−6	−5	−4	−3	−2	−1	0
Observed	1	1	4	4	3	7	18	21	38	78	83
Calc. I	0	0	0	1	3	9	20	37	58	76	83
O–C	1	1	4	3	0	−2	−2	−16	−20	2	0
Calc. II	0	0	3	2	1	5	15	18	35	74	79

x_i (sec)	1	2	3	4	5	6	7	8	9	10
Observed	68	51	38	26	15	10	8	4	6	0
Calc. I	76	58	37	20	9	3	1	0	0	0
O–C	−8	−7	1	6	6	7	7	4	6	0
Calc. II	64	46	33	21	9	4	2	0	0	0

The mode of the distribution is approximately zero so an estimate of the standard error, σ, can be obtained from the ratio of the number of residuals between $\pm 4{\cdot}5$ s to the number between $\pm 2{\cdot}5$ s. The probability that x_i is between $\pm a$ is erf ha (Jeffreys 1939c, p. 61) so that

$$\frac{\operatorname{erf} 4{\cdot}5h}{\operatorname{erf} 2{\cdot}5h} = \frac{421}{318} = 1{\cdot}32,$$

where $2h^2\sigma^2 = 1$.

The solution is $h = 0{\cdot}30$ so that computing $83 \exp(-h^2x_i^2)$ we obtain the frequencies called Calc. I in Table 4. The differences (O − C) suggest that the asymmetry can be dealt with by a uniform reduction of $4+\frac{1}{3}x_i$. This gives Calc. II in Table 4, with $h = 0{\cdot}29$ and $\sigma = 2{\cdot}4$ s. Adopting this approximation, the mean for each range, shown in Table 5, was then calculated using appropriate weights.

Table 5

Means and unsmoothed travel-times for PKIKP

Mean distance (°)	Mean residual s	Unsmoothed travel-times min	s
112·6	+1·49 ± 0·55	18	39·8
117·6	+1·98 ± 0·41		50·0
122·6	+1·44 ± 0·25		59·1
127·6	+1·11 ± 0·23	19	8·5
132·6	+1·41 ± 0·24		18·3
137·6	+2·11 ± 0·29		28·2

These means were used as corrections to the J.–B. *PKIKP* tables to give the "Unsmoothed travel-times" in Table 5. These were first fitted to the form

$$t = 19\,\text{min}\ 3\,\text{s} + 9x + a + bx + cx^2,$$

where $5x = \Delta - 125°$. The formal solution is

$$a = 0{\cdot}57 \pm 0{\cdot}17;\ b = 0{\cdot}53 \pm 0{\cdot}03;\ c = 0{\cdot}09 \pm 0{\cdot}02. \qquad (1)$$

By theory, if all included observations are *PKIKP*, *DE* is curved downwards and c is negative, so that this solution presents some difficulties in interpretation. (The upward curvature may disappear if the present data were combined with similar observations for $\Delta > 145°$.) Since the quadratic term in (1) contributes a maximum of only 0·8 s at 110° and 140°, we alternatively fit the times to a linear form, yielding

$$t = 19\,\text{min}\ 3{\cdot}7\,\text{s} + 1{\cdot}92(\Delta - 125°)\text{s}, \qquad (2)$$

with $\sigma = 0{\cdot}6$ s.

For $110° < \Delta < 140°$, the mean difference between (2) and the J.–B. tables for the *DE* branch of *PKP* is +1·5 s. This agrees with the mean *PKIKP* residual of 1·2 s for $\Delta < 140°$ in Jeffreys's preferred solution (1942, p. 36) of the 1934 June 29 earthquake.

5. Travel-times for $135° < \Delta < 145°$ for the early observations

Some of the residuals for $\Delta > 140°$ may belong to *PKIKP*. To reduce the effect of these, consideration was limited to the residuals in the curly brackets in Tables 2 and 3. After subtraction of rough means in each range a total distribution was formed which, apart from a slight scarcity of observations in the central frequency, resembled a normal one with $\sigma \approx 1{\cdot}9$, superimposed on a uniform background of 2 s.

Proceeding as in Section 4, seven weighted means were used to obtain, for $135°{\cdot}0 < \Delta < 145°{\cdot}0$, the linear form

$$t = 19\,\text{min}\,(29{\cdot}1 \pm 0{\cdot}12)\text{s} + (2{\cdot}69 \pm 0{\cdot}03)(\Delta - 141°{\cdot}6)\text{s}, \qquad (3)$$

with $\sigma = 1{\cdot}19$ s.

For $143°{\cdot}0 < \Delta < 145°{\cdot}0$, the J.–B. times for the branch *BC* are within 1 s of the times to the same distance given by (3) and the gradient of (3) equals the J.–B. gradient near 145°. This suggests that (3) corresponds to the *BC* branch of *PKP* and its extension for diffracted waves.

It is of interest that (3) agrees with Jeffreys's preferred solution of the 1934 June 29 shock. This gives residuals against (3) of +1, 0, and 0 s at Pittsburgh (140°·8), Oak Ridge (142°·2) and Fordham (142°·9), respectively.

6. Discussion

The equations (2) and (3) are compatible with *PKP* observations from four 1954 hydrogen bomb explosions (Burke-Gaffney & Bullen 1958). (The mean *P* residual calculated from data given by Bullen & Burke-Gaffney (1958), is $\delta = -2{\cdot}7 \pm 0{\cdot}3$ s from 17 observations with $69° < \Delta < 97°$.)

First, allowing for δ, four readings at Stuttgart at 116°·2 give a mean residual of +0·2 s against equation (2). Secondly, the small first arrivals at Pretoria,

Kimberley and Tamanrasset (Bullen & Burke-Gaffney *loc. cit.*) give the following mean residuals against equation (3) after the adjustment δ is applied. (n is the number of observations.)

	Distance	n	O–C
	°		s
Pretoria	137·3	3	−0·2
Kimberley	139·5	4	+0·5
Tamanrasset	140·4	4	+1·9

There is a suggestion of a systematic deviation but agreement between the observations and (3) is well within the uncertainties. (Bullen & Burke-Gaffney remark that the uncertainty of the Tamanrasset readings is about 3 s.)

Recent investigations of the *PKP* phase have also been made by Gutenberg (1951, 1957, 1958a, 1958b) and Denson (1952). They show distributions of early short-period *PKP* phases for $\Delta < 145°$ which are considered by Gutenberg to be a consequence of dispersion near the boundary of the inner core. The present data, which take no account of wave periods, provide no evidence of multiplication of *PKP* branches. The derived travel-time curves (2) and (3) have very similar gradients to the J.–B. *DE* and *BC* curves respectively, and the standard errors have magnitudes usual in this type of study (Jeffreys 1939b). The results may be explained in terms of refraction at the boundary of the inner core and diffraction near the *PKP* caustic at *B*. They support Jeffreys's suggestion (1942) that the radius of the inner core derived from the J.–B. tables (~ 1250 km) needs a slight reduction.

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University of Sydney,
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The External Gravity Field of a Rotating Spheroid to the order of e^3

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Summary

The theory of the external gravity field of a rotating body bounded by a spheroidal equipotential surface is carried to terms of the order of e^3. The potential is expanded in functions that are orthogonal in the space outside an oblate spheroid. It is found that terms of order e^3 in the gravity formula and in the relation between J, e and m are mostly very small.

1. The problems

The theory of the variation of gravity over a rotating spheroid on which the International Gravity Formula is based is taken as far as terms of order e^2 where e is the ellipticity of the surface. The relation between the constants J, e and m, which connects the effect of the Earth on the motion of the Moon and artificial satellites with the variation of gravity at the surface of the Earth, is also carried as far as terms of order e^2. Hitherto this approximation has been quite adequate but it is now clear that the potential accuracy of observations on artificial satellites requires that terms of order e^3 should be examined. The same order of approximation will be required if the distribution of gravity observations over the surface of the Earth should be improved so that the amplitudes of the lower order harmonic terms can be determined to 1 mgal. This is by no means unlikely. Accordingly, in this paper the formula for surface gravity and the relation between J, e and m are taken to order e^3. The Earth is not, of course, an exact spheroid but its external gravity field departs only very slightly from that of an exact spheroid, and so the theory of the external field of an exact spheroid provides an excellent standard of comparison for the field of the Earth itself.

The formulae have in the past been derived from an expansion of the potential outside a spheroid in solid spherical harmonics but these functions are not orthogonal for the domain exterior to a spheroid and the nature of the approximations involved would require examination. As is well known, the theory of surface gravity has also been developed (Pizetti 1894) using a class of functions, elliptic functions, that are orthogonal outside an ellipsoid, and the formulae, which are exact, may be expanded to terms in e^3. However, this method is less convenient for the study of the external field and since the spherical harmonic method for finding the relation between J, e and m also requires investigation, it is convenient to use instead a second class of functions (Spheroidal harmonics) involving Legendre functions, orthogonal outside an oblate spheroid. (See Jung 1956). As a check

on the algebra the gravity formula is established by the use of this second class of functions as well as from Pizetti's formula. An incidental advantage of using spheroidal harmonics is that the reason for the existence of a fourth harmonic term (and higher ones) in the spherical harmonic expansion of gravity outside a spheroid is clarified. Mineo (1930, 1933) has discussed the accuracy of Clairaut's and Stokes's theorems when derived by the use of spherical harmonics.

The plan of this paper is as follows:

The coordinate system and some necessary formulae are set out in Section 2. Section 3 contains the derivation of the gravity formula over a spheroid from the sole condition that it is an equipotential surface of the sum of the gravitational and rotational potentials and in Section 4 the relation between the ellipticity e and the mass and moments of inertia of the rotating spheroid is obtained by direct integration without recourse to MacCullagh's theorem. The expansion of the potential at external points, required in the theory of the motion of satellites, is derived in Section 5.

It is hoped in a later paper to derive the normal gradient of gravity and to extend Stokes's theorem for the determination of the elevation of the geoid from gravity anomalies to terms of order e.

2. The coordinate system

We take orthogonal coordinates $(\eta, \vartheta, \lambda)$ defined in terms of Cartesian coordinates as follows:

$$x = c \cosh\eta \sin\vartheta \cos\lambda,$$

$$y = c \cosh\eta \sin\vartheta \sin\lambda,$$

$$z = c \sinh\eta \cos\vartheta.$$

We assume that there is rotational symmetry about the axis of rotation z.

The surfaces η = constant are confocal spheroids and the surfaces ϑ = constant are confocal hyperboloids of one sheet. The surfaces λ = constant are meridional planes. ϑ is the eccentric angle and λ the longitude.

We suppose that all matter is contained within the surface $\eta = \eta_0$ and that this surface is an equipotential surface for the geopotential (surfaces with other values of η are not equipotentials).

The major and minor axes of the spheroid bounded by this surface are respectively:

$$a = c \cosh\eta_0$$

and

$$b = c \sinh\eta_0.$$

If $e = (a-b)/a$, it is readily shown that

$$\left.\begin{aligned} &\cosh\eta_0 = \{e(2-e)\}^{-\frac{1}{2}} \\ \text{and} \quad &\sinh\eta_0 = (1-e)\{e(2-e)\}^{-\frac{1}{2}}. \end{aligned}\right\} \qquad (2.1)$$

For this coordinate system, the line element ds is given by:

$$ds^2 = h_1^2\, d\eta^2 + h_2^2\, d\vartheta^2 + h_3^2\, d\lambda^2,$$

where

$$h_1^2 = c^2(\sinh^2\eta + \cos^2\vartheta),$$
$$h_2^2 = c^2(\sinh^2\eta + \cos^2\vartheta)$$

and

$$h_3^2 = c^2 \cosh^2\eta \sin^2\vartheta.$$

The surface element is

$$dS = h_2 h_3 \, d\vartheta \, d\lambda$$
$$= c^2(\sinh^2\eta + \cos^2\vartheta)^{\frac{1}{2}} \cosh\eta \sin\vartheta \, d\vartheta \, d\lambda.$$

The element of volume is

$$d\tau = h_1 h_2 h_3 \, d\eta \, d\vartheta \, d\lambda$$
$$= c^3(\sinh^2\eta + \cos^2\vartheta) \cosh\eta \sin\vartheta \, d\eta \, d\vartheta \, d\lambda.$$

The gradient of a function which is constant on the surface η is $\partial/h_1\partial\eta$, that is

$$\frac{1}{c(\sinh^2\eta + \cos^2\vartheta)^{\frac{1}{2}}} \frac{\partial}{\partial\eta}.$$

The expansion of $(\sinh^2\eta_0 + \cos^2\vartheta)^{-\frac{1}{2}}$ will often be required:

$$(\sinh^2\eta_0 + \cos^2\vartheta)^{-\frac{1}{2}} = \frac{1}{\sinh\eta_0}\left\{1 - \frac{1}{2}\frac{\cos^2\vartheta}{\sinh^2\eta_0} + \frac{3}{8}\frac{\cos^4\vartheta}{\sinh^4\eta_0} - \frac{5}{16}\frac{\cos^6\vartheta}{\sinh^6\eta_0}\right\}$$

to order e^3, that is

$$\frac{1}{\sinh\eta_0}\{1 - e(1 + \frac{3}{2}e + 2e^2)\cos^2\vartheta + \frac{3}{2}e^2(1 + 3e)\cos^4\vartheta - \frac{5}{2}e^3\cos^6\vartheta\}. \qquad (2.2)$$

Finally, we relate ϑ to the geographical latitude ϕ. In the meridian $\lambda = 0$

$$\cot\phi = \frac{\partial x}{\partial\eta}\Big/\frac{\partial z}{\partial\eta} \text{ evaluated at } \eta = \eta_0$$
$$= (1 - e)\tan\vartheta.$$

Then

$$\cos^2\vartheta = \{1 - (2e - e^2)\cos^2\phi - 4e^2\sin^2\phi\cos^2\phi\}\sin^2\phi. \qquad (2.3)$$

In terms of geographical latitude, $(\sinh^2\eta_0 + \cos^2\vartheta)^{-\frac{1}{2}}$ is equal to

$$\frac{1}{\sinh\eta_0}\{1 - e\sin^2\phi + \tfrac{1}{8}e^2\sin^2 2\phi + \tfrac{1}{8}e^3\sin^2\phi\sin^2 2\phi\}. \qquad (2.4)$$

3. The gravity formula

The orthogonal solutions of Laplace's equation for the space external to the surface $\eta = \eta_0$ which are symmetrical about the z-axis are

$$-v_n \frac{Q_n(i\sinh\eta)}{Q_n(i\sinh\eta_0)} P_n(\cos\vartheta),^*$$

* The minus sign preserves consistency with other expressions for the gravitational potential.

(Hobson 1931, p. 431) where the Legendre functions of the second kind are defined by

$$Q_n(\mu) = \frac{1}{2}\int_{-1}^{+1} \frac{P_n(\mu)}{\mu - u}\, du.$$

On the surface η_0 the gravitational potential, V, of the mass inside η_0 is accordingly

$$-\Sigma v_n P_n(\cos\vartheta),$$

while the rotational potential is

$$\tfrac{1}{2}\omega^2 c^2 \cosh^2\eta_0 (1 - \cos^2\vartheta);$$

ω is the angular velocity of the Earth about the z-axis.

Since the sum of the gravitational and rotational potentials has to be constant over the surface $\eta = \eta_0$, it follows that

$$v_2 = \tfrac{1}{3}\omega^2 c^2 \cosh^2\eta_0 = \tfrac{1}{3}\omega^2 a^2 \tag{3.1}$$

while all the other coefficients except v_0 vanish.

It may be noted here that when expressed in this coordinate system the gravitational potential over a spheroidal equipotential surface has no fourth harmonic term although it does have one when expressed in spherical polar coordinates. The fourth harmonic in the potential of a spheroid is thus in a sense fictitious since it may be removed by a suitable coordinate transformation.

The total potential U on or outside the surface η_0 may then be written as

$$-v_0 \frac{Q_0(i\sinh\eta)}{Q_0(i\sinh\eta_0)} + \frac{\omega^2 a^2}{3}\frac{Q_2(i\sinh\eta)}{Q_2(i\sinh\eta_0)} P_2(\cos\vartheta) + \tfrac{1}{2}\omega^2 c^2 \cosh^2\eta(1 - \cos^2\vartheta). \tag{3.2}$$

The value of gravity, g, is the modulus of the derivative of U normal to the surface η_0, that is

$$\left| \frac{i\cosh\eta}{c(\sinh^2\eta + \cos^2\vartheta)^{\frac{1}{2}}} \frac{\partial U}{\partial(i\sinh\eta)} \right| \tag{3.3}$$

evaluated at $\eta = \eta_0$.

It is convenient first to determine the constant v_0. By Gauss's theorem

$$\int_S \operatorname{grad} V . \mathbf{dS} = 4\pi f M,$$

where M is the mass of the spheroid, f is the Newtonian constant and the integral is taken over the surface η_0.

Now

$$\operatorname{grad} V . dS = \frac{h_2 h_3\, d\vartheta\, d\lambda}{c(\sinh^2\eta + \cos^2\vartheta)^{\frac{1}{2}}} \frac{\partial V}{\partial \eta}$$

$$= ci\cosh^2\eta \sin\vartheta\, d\vartheta\, d\lambda \frac{\partial V}{\partial(i\sinh\eta)}.$$

The integral with respect to λ gives a factor 2π while that with respect to ϑ eliminates the term with $P_2(\cos\vartheta)$.

Hence

$$\int_S \operatorname{grad} V \,.\, \mathbf{dS} = 4\pi v_0 ic \cosh^2\eta_0 \left.\frac{Q_0'}{Q_0}\right|_{\eta_0},$$

where

$$Q_0' = \frac{dQ_0(i\sinh\eta)}{d(i\sinh\eta)}.$$

But

$$Q_n(\mu) = \frac{\pi^{\frac{1}{2}} n!}{2^{n+1}(n+\frac{1}{2})!}\frac{1}{\mu^{n+1}} F\left(\frac{n+1}{2}, \frac{n+2}{2}; \frac{2n+3}{2}; \mu^{-2}\right)$$

where F is the hypergeometric function (Hobson, 1931, pp. 14 and 53).

Hence

$$Q_0 = \frac{1}{\mu}\left(1 + \frac{1}{3\mu^2} + \frac{1}{5\mu^4} + \frac{1}{7\mu^6}\right)$$

and

$$Q_0' = 1/\cosh^2\eta.$$

Accordingly

$$\int_S \operatorname{grad} V \,.\, \mathbf{dS} = 4\pi v_0 ic\, Q_0(i\sinh\eta_0)$$

and

$$v_0 = -\frac{fM}{ic} Q_0(i\sinh\eta_0) \tag{3.4}$$

$$= \frac{fM}{b}\left(1 - \frac{1}{3\sinh^2\eta_0} + \frac{1}{5\sinh^4\eta_0} - \frac{1}{7\sinh^6\eta_0}\right)$$

$$= \frac{fM}{a}\left(1 + \frac{1}{3}e + \frac{2}{15}e^2 + \frac{2}{35}e^3\right). \tag{3.4a}$$

The product

$$v_0 \left.\frac{Q'_0}{Q_0}\right|_{\eta_0}$$

will be required; it is

$$\frac{fM}{a}\frac{1}{i\sinh\eta_0}(1-e). \tag{3.5}$$

The ratio Q_2'/Q_2 is also needed; it may be obtained from the hypergeometric form of Q_2 and is

$$-\frac{3}{i\sinh\eta_0}\left(1 - \frac{8}{7}e + \frac{20}{147}e\right) \text{ to order } e^2. \tag{3.6}$$

The value of gravity at $\eta = \eta_0$ is then the modulus of

$$\frac{i\cosh\eta_0}{c(\sinh^2\eta_0+\cos^2\vartheta)^{\frac{1}{2}}}\times\left[\frac{fM}{a}\frac{1-e}{i\sinh\eta_0}+\frac{\omega^2a^2}{i\sinh\eta_0}\left(1-\frac{8}{7}e+\frac{20}{147}e^2\right)\times\frac{3\cos^2\vartheta-1}{2}\right]-$$

$$-\frac{1}{c(\sinh^2\eta_0+\cos^2\vartheta)^{\frac{1}{2}}}\omega^2c^2\cosh\eta_0\sinh\eta_0\times(1-\cos^2\vartheta),$$

that is

$$\frac{1}{c(\sinh^2\eta_0+\cos^2\vartheta)^{\frac{1}{2}}}\left[\frac{fM}{a}-\frac{3}{2}\omega^2a^2\left(1-\frac{5}{7}e+\frac{17}{441}e^2\right)+\right.$$

$$\left.+\frac{\omega^2a^2}{2}\times\left(5-\frac{17}{7}e-\frac{1}{49}e^2\right)\cos^2\vartheta\right]. \tag{3.7}$$

The value of g at the equator, where $\cos\vartheta = 0$, is

$$g_e=\frac{1}{c\sinh\eta_0}\left[\frac{fM}{a}-\frac{3}{2}\omega^2a^2\left(1-\frac{5}{7}e-\frac{1}{441}e^2\right)\right] \tag{3.8}$$

$$=\frac{fM}{ab}-\frac{3}{2}\omega^2a\left(1+\frac{2}{7}e+\frac{125}{441}e^2\right). \tag{3.8a}$$

Now let $\omega^2a^3(1-e)/fM$ be denoted, as is usual, by m and change from ϑ to ϕ. With these substitutions,

$$g=g_e\left[1+m\left(\frac{5}{2}+\frac{9}{7}e+\frac{15}{4}m+\frac{125}{98}e^2+3em+\frac{45}{8}m^2\right)\sin^2\phi-\right.$$

$$-\frac{em}{4}\left(5+\frac{e}{14}+\frac{15}{2}m\right)\sin^2 2\phi-\frac{5}{2}e^2m\sin^2\phi\sin^2 2\phi\Big]\times$$

$$\times\left(1-e\sin^2\phi+\frac{e^2}{8}\sin^2 2\phi+\frac{e^3}{8}\sin^2\phi\sin^2 2\phi\right).$$

Thus, finally we have for the value of gravity over the spheroid, the following expression carried to the third order in e:

$$g_e\left[1+\left(\frac{5}{2}m-e+\frac{15}{4}m^2-\frac{17}{14}em-\frac{1}{98}e^2m-\frac{3}{4}em^2+\frac{45}{8}m^3\right)\sin^2\phi+\right.$$

$$\left.+\frac{1}{8}e\left(e-5m+\frac{17}{7}em-\frac{15}{2}m^2\right)\sin^2 2\phi-\frac{1}{8}e^2\left(\frac{15}{2}m-e\right)\sin^2\phi\sin^2 2\phi\right]. \tag{3.9}$$

To the second order in e and m this agrees with the usual gravity formula (Jeffreys 1952, p. 131).

At the poles, where $\cos\vartheta = 1$, the value of gravity is

$$g_p=\frac{1}{a}\left\{\frac{fM}{a}+\omega^2a^2\left(1-\frac{e}{7}\right)\right\}$$

to second order. Hence

$$\frac{g_p}{b}-\frac{g_e}{a}=\frac{5}{2}\omega^2\left(1+\frac{18}{35}e+\frac{25}{49}e^2\right) \tag{3.10}$$

(Clairaut's theorem).

The following formula will be found to be exact:

$$\frac{2g_e}{a}+\frac{g_p}{b}=4\pi f\bar{\rho}-2\omega^2 \tag{3.11}$$

where $\bar{\rho}$ is the mean density of the matter in the spheroid (Pizetti's theorem).

It may be noted that although the potential of a spheroid contains no harmonics higher than the second when expressed in $(\eta, \vartheta, \lambda)$ coordinates, the gravity formula always contains such terms because of the factor $(\sinh^2\eta_0+\cos^2\vartheta)^{-\frac{1}{2}}$.

Comparison with Pizetti's theory

Pizetti's result is expressed in the following formula

$$\frac{g}{g_e}=\frac{1+(\beta-e-e\beta)\sin^2\phi}{[1-e(2-e)\sin^2\phi]^{\frac{1}{2}}}$$

where

$$g_e=\frac{fM}{ab}-\tfrac{3}{2}\omega^2a[1+C(e)]$$

and β, which is $(g_p-g_e)/g_e$, is given by

$$\beta+e=\frac{2\omega^2a(1-e)\,C(e)}{g_e}.$$

$C(e)$ is the function

$$\frac{5}{4}\left(1+\frac{18}{35}e+\frac{125}{245}e^2\right).$$

Consider first the value of g_e. By equation (3.8a) this is

$$\frac{fM}{ab}-\frac{3}{2}\omega^2a\left(1+\frac{2}{7}e+\frac{125}{441}e^2\right)$$

which is identical with Pizetti's result.

By Pizetti's formula,

$$\beta+e=\frac{5}{2}m\left(1-\frac{17}{35}e+\frac{3}{2}m-\frac{1}{245}e^2-\frac{3}{10}em+\frac{9}{4}m^2\right)$$

or

$$\beta=\frac{5}{2}m-e-\frac{17}{14}em+\frac{15}{4}m^2-\frac{1}{98}e^2m-\frac{3}{4}em^2+\frac{45}{8}m^3.$$

Now

$$\{1-e(2-e)\sin^2\phi\}^{-\frac{1}{2}}$$
$$=1+e(1+e+e^2)\sin^2\phi-\tfrac{1}{8}e^2(3+2e)\sin^2 2\phi-\tfrac{1}{8}e^3\sin^2\phi\sin^2 2\phi$$

and so

$$\frac{g}{g_e} = 1+\beta\sin^2\phi + \frac{1}{8}e\left(e-5m+\frac{17}{7}em-\frac{15}{2}m^2\right)\sin^2 2\phi - \frac{1}{8}e^2\left(\frac{15}{2}m-e\right)\sin^2\phi\sin^2 2\phi,$$

in agreement with equation (3.9)

4. The relation between e, m and J

MacCullagh's theorem shows that in spherical polar coordinates the external potential of a body symmetrical about its polar axis may be written in the form

$$V = \frac{fM}{r}\left\{1-\frac{2Ja^3}{3r^3}P_2(\cos\theta)+\ldots\right\},$$

where the coefficient J is equal to $\frac{1}{2}(C-A)/(Ma^2)$, C being the moment of inertia about the polar axis and A that about any axis in the equatorial plane.

This result might be used to relate e, m and J by transforming to spheroidal coordinates and identifying the coefficient v_2 of the preceding formulae. However, the convergence of this particular form of MacCullagh's theorem on the boundary of the body requires discussion since the conditions for an expansion in spherical harmonics are only fulfilled on the surface of a sphere.

The relation between J, e and m will therefore be obtained directly without recourse to MacCullagh's theorem.

If d is the distance between two points with spheroidal coordinates $(\eta', \vartheta', \lambda')$ and $(\eta_0, \vartheta, \lambda)$ where $\eta_0 > \eta'$, then (Hobson 1931, p. 430)

$$\frac{c}{d} = i\sum_n (2n+1)\left[P_n(\cos\vartheta)P_n(\cos\vartheta')Q_n(i\sinh\eta_0)P_n(i\sinh\eta') + 2\sum_{m=1}^{n}(-)^m \times \right.$$

$$\left. \times \left(\frac{(n-m)!}{(n+m)!}\right)^2 P_n^m(\cos\vartheta)P_n^m(\cos\vartheta')Q_n^m(i\sinh\eta_0)P_n^m(i\sinh\eta')\cos m(\lambda-\lambda')\right].$$

This formula can be applied over the surface of a spheroid where the expansion in solid spherical harmonics does not necessarily converge.

The potential of the body at a point on η_0 is

$$f\int_\tau \frac{\rho\, d\tau}{d}$$

and if the body has rotational symmetry, as we assume throughout, then this is equal to

$$\frac{fi}{c}\sum_n (2n+1)P_n(\cos\vartheta)Q_n(i\sinh\eta_0)\int_\tau d\tau\,\rho P_n(\cos\vartheta')P_n(i\sinh\eta').$$

The coefficients of the successive terms in the expansion of V on the surface η_0 are thus

$$\frac{fi}{c}(2n+1)Q_n(i\sinh\eta_0)\int_\tau d\tau\,\rho P_n(\cos\vartheta')P_n(i\sinh\eta').$$

For $n=0$, the coefficient is

$$\frac{fM}{c}.iQ_0(i\sinh\eta_0)$$

as found previously.

For $n=2$, the coefficient v_2 is

$$\frac{5f}{c}iQ_2(i\sinh\eta_0)\int_\tau d\tau\,\rho P_2(\cos\vartheta')P_2(i\sinh\eta').$$

To interpret this we require an expression for $C-A$. Taking A to be the moment about the x-axis,

$$C=\int d\tau\,\rho(x^2+y^2)$$

and

$$A=\int d\tau\,\rho(y^2+x^2).$$

We have

$$d\tau = h_1h_2h_3\,d\eta'\,d\vartheta'\,d\lambda',$$
$$x^2+y^2 = c^2\cosh^2\eta'\sin^2\vartheta'$$

and

$$y^2+z^2 = c^2(\cosh^2\eta'\sin^2\vartheta'\sin^2\lambda'+\sinh^2\eta'\cos^2\vartheta').$$

We write $h_1h_2h_3=h^3$ and then, carrying out the integration with respect to λ', we find that

$$C-A = c^2\pi\int d\eta'\int d\vartheta'\,h^3\rho(\cosh^2\eta\sin^2\vartheta'-2\sinh^2\eta'\cos^2\vartheta').$$

But $\cosh^2\eta'\sin^2\vartheta'-2\sinh^2\eta'\cos^2\vartheta'$ is equal to

$$\tfrac{2}{3}\{1+2P_2(i\sinh\eta')P_2(\cos\vartheta')\}$$

and M is equal to

$$2\pi\int d\eta'\int d\vartheta'\,h^3\rho.$$

Hence

$$C-A=\frac{c^2M}{3}+\frac{2}{15}c^3\frac{v_2}{ifQ_2(i\sinh\eta_0)}.$$

J is defined as $\frac{2}{3}(C-A)/(Ma^2)$. Then since $v_2=\frac{1}{3}\omega^2a^2$,

$$\frac{2}{3}JMa^2=\frac{1}{3}Mc^2+\frac{2}{15}c^3\frac{\omega^2a^2}{3ifQ_2(i\sinh\eta_0)}.$$

Now

$$\frac{\omega^2 a^3(1-e)}{fM} = m$$

and

$$iQ_2(i\sinh\eta_0) = -\frac{2}{15}\frac{1}{\sinh^3\eta_0}\left(1-\frac{12}{7}e+\frac{2}{7}e^2\right)$$

and accordingly

$$J = \frac{1}{2\cosh^2\eta_0} - \frac{m}{2(1-e)}\left(\frac{\sinh\eta_0}{\cosh\eta_0}\right)\left(1-\frac{12}{7}e+\frac{2}{7}e^2\right)^{-1}$$

that is

$$J = e\left(1-\frac{1}{2}e\right) - \frac{1}{2}m\left(1-\frac{2}{7}e+\frac{11}{49}e^2\right) \text{ to order } e^3. \tag{4.1}$$

This result agrees with that hitherto established to order e^2.

5. The gravitational potential as a function of spherical polar coordinates

For use in the theory of the orbits of satellites, the gravitational potential is required as a function of spherical polar coordinates. The transformation from spheroidal coordinates (η, ϑ) to spherical polars (r, θ) is first required.

We have

$$r^2 = c^2(\sinh^2\eta + \sin^2\vartheta)$$

and

$$\cos\theta = (c/r)\sinh\eta\cos\vartheta.$$

Put $\sinh\eta = s^{-1}$, $\sigma = c/r$ and $\rho = a/r$ so that $\sigma^2 = e(2-e)\rho^2$.

Then

$$s^4\cos^2\theta + s^2(1-\sigma^2) - \sigma^2 = 0,$$

whence

$$\frac{s^2}{\sigma^2} = 1+\sigma^2(1-\cos^2\theta)+\sigma^4(1-3\cos^2\theta+2\cos^4\theta)+ \\ +\sigma^6(1-6\cos^2\theta+10\cos^4\theta-5\cos^6\theta).$$

The following are also required:

$$\frac{s}{\sigma} = 1+\tfrac{1}{2}\sigma^2(1-\cos^2\theta)+\tfrac{1}{8}\sigma^4(3-10\cos^2\theta+7\cos^4\theta)+ \\ +\tfrac{1}{16}\sigma^6(5-35\cos^2\theta+63\cos^4\theta-33\cos^6\theta);$$

$$\left(\frac{s}{\sigma}\right)^3 = 1+\tfrac{3}{2}\sigma^2(1-\cos^2\theta)+\tfrac{3}{8}\sigma^4(5-14\cos^2\theta+9\cos^4\theta).$$

Also

$$\cos^2\vartheta = \frac{s^2}{\sigma^2}\cos^2\theta.$$

Let the gravitational potential V be written as $-(V_0+V_2)$ where

$$V_0 = v_0\frac{Q_0(i\sinh\eta)}{Q_0(i\sinh\eta_0)}$$

and

$$V_2 = \tfrac{1}{3}\omega^2a^2\frac{Q_2(i\sinh\eta)}{Q_2(i\sinh\eta_0)}P_2(\cos\vartheta).$$

We take V_0 first. From equation 3.4,

$$v_0 = -\frac{fM}{ic}Q_0(i\sinh\eta_0)$$

and so

$$\begin{aligned} V_0 &= -\frac{fM}{ic}Q_0(i\sinh\eta) \\ &= \frac{fM}{c\sinh\eta}\left[1-\frac{1}{3\sinh^2\eta}+\frac{1}{5\sinh^4\eta}-\frac{1}{7\sinh^6\eta}\right] \\ &= fM\frac{s}{c}\left[1-\frac{s^2}{3}+\frac{s^4}{5}-\frac{s^6}{7}\right]. \end{aligned}$$

With the expansions just found

$$\frac{s}{c} = \frac{1}{r}\{1+\tfrac{1}{2}\sigma^2(1-\cos^2\theta)+\tfrac{1}{8}\sigma^4(3-10\cos^2\theta+7\cos^4\theta)+\tfrac{1}{16}\sigma^6(5-35\cos^2\theta+ \\ +63\cos^4\theta-33\cos^6\theta)\}$$

and

$$1-\frac{s^2}{3}+\frac{s^4}{5}-\frac{s^6}{7} = 1-\frac{1}{3}\sigma^2-\frac{1}{15}\sigma^4(2-5\cos^2\theta)- \\ -\frac{1}{105}\sigma^6(8-63\cos^2\theta+70\cos^4\theta).$$

Hence

$$\frac{V_0r}{fM} = 1-\frac{1}{3}\sigma^2P_2(\cos\theta)+\frac{1}{5}\sigma^4P_4(\cos\theta)-\frac{1}{7}\sigma^6P_6(\cos\theta).$$

To obtain V_2 we require the ratio

$$\frac{Q_2(i\sinh\eta)}{Q_2(i\sinh\eta_0)}$$

which is

$$\left(\frac{\mu_0}{\mu}\right)^3\frac{F(\frac{3}{2},2;\frac{7}{2};\mu^{-2})}{F(\frac{3}{2},2;\frac{7}{2};\mu_0^{-2})},$$

where $\mu = i \sinh \eta$, $\mu_0 = i \sinh \eta_0$,

$$\left(\frac{\mu_0}{\mu}\right)^3 = -(1-e)^3 (s/\sigma)^3 \rho^3$$

and $(s/\sigma)^3$ has been given above.

$$F\left(\frac{3}{2}, 2; \frac{7}{2}; \mu_0^{-2}\right) = 1 - \frac{12}{7}e + \frac{2}{7}e^2$$

and

$$F\left(\frac{3}{2}, 2; \frac{7}{2}; \mu^{-2}\right) = 1 - \frac{6}{7}s^2 + \frac{5}{7}s^4$$
$$= 1 - \frac{6}{7}\sigma^2 - \frac{1}{7}\sigma^4(1 - 6\cos^2\theta).$$

Combining these results,

$$\left(\frac{r}{a}\right)^3 \frac{1}{1-e} \frac{Q_2(i\sinh\eta)}{Q_2(i\sinh\eta_0)} = -\left(1 - \frac{2}{7}e + \frac{11}{49}e^2\right) \times$$
$$\times \left\{1 + \frac{3}{2}\sigma^2\left(\frac{3}{7} - \cos^2\theta\right) + \frac{1}{56}\sigma^4(25 - 174\cos^2\theta + 189\cos^4\theta)\right\}.$$

Using the expression for $\cos\vartheta$ given above,

$$P_2(\cos\vartheta) = \tfrac{1}{2}(3\cos^2\theta - 1) + \tfrac{3}{2}\sigma^2(1 - \cos^2\theta)\cos^2\theta$$
$$+ \tfrac{3}{8}\sigma^4(1 - 3\cos^2\theta + 2\cos^4\theta)\cos^2\theta.$$

Hence,

$$V_2 = \frac{1}{3}\frac{fM}{a}\frac{m}{1-e}\frac{Q_2(i\sinh\eta)}{Q_2(i\sinh\eta_0)}P_2(\cos\vartheta)$$
$$= \frac{1}{3}\frac{fM}{a}m\left(1 - \frac{2}{7}e + \frac{11}{49}e^2\right)\left(\frac{a}{r}\right)^3 \times$$
$$\times \left\{P_2(\cos\theta) - \frac{6}{7}\sigma^2 P_4(\cos\theta) + \frac{5}{7}\sigma^4 P_6(\cos\theta)\right\}.$$

The complete expression for the potential is thus

$$\frac{fM}{r}\left[1 - \frac{1}{3}\left\{\sigma^2 - \left(\frac{a}{r}\right)^2 m\left(1 - \frac{2}{5}e + \frac{11}{49}e^2\right)\right\}P_2(\cos\theta) +\right.$$
$$\left. + \sigma^2\left\{\frac{1}{5}\sigma^2 - \frac{2}{7}\left(\frac{a}{r}\right)^2 m\left(1 - \frac{2}{7}e\right)\right\} \times P_4(\cos\theta) - \sigma^4\left\{\frac{\sigma^2}{7} - \frac{5}{21}\left(\frac{a}{r}\right)^4 m\right\}P_6(\cos\theta)\right].$$

Since $\sigma^2 = e(2-e)(a/r)^2$, we have finally,

$$V = \frac{fM}{r}\left[1 - \frac{2}{3}\left(\frac{a}{r}\right)^2 J P_2(\cos\theta) + \frac{8}{35}\left(\frac{a}{r}\right)^4 D P_4(\cos\theta) + \left(\frac{a}{r}\right)^6 J_6 P_6(\cos\theta)\right],$$

where

$$J = e\left(1 - \frac{1}{2}e\right) - \frac{1}{2}m\left(1 - \frac{2}{7}e + \frac{11}{49}e^2\right), \qquad (5.1a)$$

as found in Section 4;

$$D = \frac{1}{2}e\left(1-\frac{1}{2}e\right)\left\{7e\left(1-\frac{1}{2}e\right)-5m\left(1-\frac{2}{7}e\right)\right\}, \tag{5.1b}$$

which again agrees to order e^2 with Darwin's result; and

$$J_6 = -\frac{4}{21}e^2(6e-5m). \tag{5.1c}$$

The usual notation in which $-\frac{2}{3}J$ is the coefficient of the second harmonic and $\frac{8}{35}D$ is the coefficient of the fourth becomes inconvenient when other harmonics have to be considered and it is therefore proposed that the potential should be written as

$$\frac{fM}{r}\left\{1-\sum_{n=2}^{\infty}J_n\left(\frac{a}{r}\right)^n P_n(\cos\theta)\right\}.$$

In this notation,

$$\begin{aligned} J_2 &= \frac{2}{3}J = \frac{2}{3}\left\{e\left(1-\frac{1}{2}e\right)-\frac{1}{2}m\left(1-\frac{2}{7}e+\frac{11}{49}e^2\right)\right\} \\ J_4 &= -\frac{8}{35}D = -\frac{4}{35}e\left(1-\frac{1}{2}e\right)\left\{7e\left(1-\frac{1}{2}e\right)-5m\left(1-\frac{2}{7}e\right)\right\} \\ J_6 &= \frac{4}{21}e^2(6e-5m). \end{aligned} \tag{5.2}$$

6. Formulae in terms of mean radius and gravity

de Sitter pointed out some time ago the advantages of using the mean radius and gravity in place of the equatorial radius and gravity on the equator as parameters in the expressions for the external potential and surface gravity. The point is that most measurements of geodetic arcs and of gravity are made in middle latitudes so that it is the mean values that are most directly determined by observation and not the equatorial values. The force of this argument is greater nowadays when the distance of the Moon is being observed by radar from stations in middle latitudes. Accordingly, the principal formulae are re-cast in terms of these parameters.

For this purpose, the exact specification of the mean must be considered. The radius and gravity may be expressed as functions of eccentric angle ϑ, of geocentric co-latitude θ, or of geographical latitude ϕ, and the mean values of these expressions differ in the terms of order e^2 and higher. For practical purposes it may be convenient to take means over geographical latitude but for theoretical purposes it seems preferable to take means over the angle ϑ; this is done here.

The radius vector r of a point on the surface of the spheroid $\eta = \eta_0$ is given by

$$\frac{r^2}{c^2} = \cosh^2\eta_0 - \cos^2\vartheta$$

so that, remembering that $c \cosh \eta_0 = a$,

$$\frac{r}{a} = 1 - \frac{1}{2}\frac{\cos^2\vartheta}{\cosh^2\eta_0} - \frac{1}{8}\frac{\cos^4\vartheta}{\cosh^4\eta_0} - \frac{1}{16}\frac{\cos^6\vartheta}{\cosh^6\eta_0}.$$

On expressing this in terms of $P_2(\cos\vartheta)$, $P_4(\cos\vartheta)$ and $P_6(\cos\vartheta)$ and using the property that the mean values of Legendre functions are zero, it follows that

$$\frac{r_m}{a} = 1 - \frac{1}{3}e + \frac{1}{15}e^2 + \frac{1}{35}e^3. \tag{6.1}$$

r_m is the required mean radius.

The external gravitational potential, V, expressed in spherical polar coordinates is

$$\frac{fM}{R} - \frac{f}{R^3}(C-A)\,P_2(\cos\theta) + \ldots$$

If this is re-written in terms of r_m, we have

$$V = \frac{fM}{r_m}\left\{\frac{r_m}{R} - \left(\frac{r_m}{R}\right)^3 \frac{C-A}{Mr_m^2} P_2(\cos\theta) + \ldots\right\}.$$

The ratio $3(C-A)/2Mr_m^2$ will be called J^*.

$$\frac{J^*}{J} = \left(\frac{a}{r_m}\right)^2.$$

Using equation (4.1),

$$J^* = e\left(1 + \frac{1}{6}e - \frac{2}{15}e^2\right) - \frac{1}{2}m\left(1 + \frac{8}{21}e + \frac{172}{735}e^2\right). \tag{6.2}$$

To obtain the gravity formula, equation (3.7) is, with the use of equation (2.2), written as

$$g = \frac{fM}{ab}\left[1 - \frac{3}{2}m\left(1 + \frac{2}{7}e + \frac{143}{441}e^2\right) + \frac{1}{2}m\left(5 + \frac{18}{7}e + \frac{143}{49}e^2\right)\cos^2\vartheta\right] \times$$
$$\times \left[1 - \frac{1}{2}e(2 + 3e + 4e^2)\cos^2\vartheta + \frac{3}{2}e^2(1 + 3e)\cos^4\vartheta - \frac{5}{2}e^3\cos^6\vartheta\right].$$

By expressing this in terms of Legendre functions of $\cos\vartheta$ and taking the constant parts, the value of g_m is obtained:

$$g_m = \frac{fM}{ab}\left(1 - \frac{2}{3}m - \frac{1}{3}e - \frac{1}{5}e^2 - \frac{7}{20}e^2m + \frac{22}{105}e^3\right),$$

or, in terms of the mean radius,

$$g_m = \frac{fM}{r_m^2}\left(1 - \frac{2}{3}m + \frac{4}{15}e^2 - \frac{2}{9}em + \frac{73}{135}e^3 - \frac{397}{540}e^2m\right). \tag{6.3}$$

It is an advantage that this relation does not involve e but only higher powers.

The complete gravity formula is

$$\frac{g}{g_m} = 1 + \left[\frac{5}{3}m - \frac{2}{3}e + \frac{10}{9}m^2 + \frac{34}{63}em - \frac{23}{63}e^2 + \frac{20}{27}m^3 + \frac{15}{14}e^2m + \frac{46}{63}em^2 + \frac{421}{945}e^3\right] \times$$
$$\times\, {}^{D}_{2}(\cos\vartheta) + \left[\frac{12}{35}e^2 - \frac{4}{7}em + \frac{982}{1155}e^2m - \frac{8}{27}em^2 + \frac{4}{11}e^3\right] P_4(\cos\vartheta) +$$
$$+ \frac{8}{231}(3e^2m - 5e^3)P_6(\cos\vartheta). \qquad (6.4)$$

7. Summary

The terms of order e^3 in the gravity formula derived in section 3 are

$$g_e\left[m\left(+\frac{45}{8}m^2 - \frac{3}{4}em - \frac{1}{98}e^2\right)\sin^2\phi + \frac{1}{8}em\left(\frac{17}{7}e - \frac{15}{2}m\right)\sin^2 2\phi\right.$$
$$\left. - \frac{1}{8}e^2\left(\frac{15}{2}m - e\right)\sin^2\phi\sin^2 2\phi\right].$$

With $e = 0{\cdot}003\,37$ and $m = 0{\cdot}003\,45$, the coefficients of the terms are respectively:

of $\sin^2\phi$	$+0{\cdot}24$ mgal;
of $\sin^2 2\phi$	$-0{\cdot}02$ mgal;
of $\sin^2\phi\sin^2 2\phi$	$+0{\cdot}03$ mgal.

It is improbable that the external field will be analysed to this accuracy for some time to come.

The formula for J is

$$e\left(1 - \frac{1}{2}e\right) - \frac{1}{2}m\left(1 - \frac{2}{7}e + \frac{11}{49}e^2\right).$$

The relative magnitude of the final term is

$$\frac{1}{2}m\frac{11}{49}e^2/J,$$

that is about $2{\cdot}5 \times 10^{-6}$.

The relative precision that could be achieved at present in the determination of J from satellite observations is about 1×10^{-4}.

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Tidal Friction from Currents near the Seabed

H. Charnock

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Summary

The current in the lowest two metres of a tidal stream has been found to increase linearly with the logarithm of the height above the bottom. Independent measurements of the shearing stress on the seabed allow a determination of von Karman's constant which was found to be approximately 0·4.

1. Introduction

Estimates of the shearing stress of a tidal current on the seabed have been published by Bowden & Fairbairn (1952, 1956). The earlier paper represents the outcome of attempts made over many years to determine the drag on the bottom using a dynamical method. It is made clear that this method is technically difficult and time-consuming and that the results are difficult to interpret. The observations described in this paper were made to supplement their investigation and to establish a simpler but less direct method.

From observations of shear flow in the laboratory, it has been confirmed that the velocity profile close, but not too close, to a boundary can be represented by the well-known expression

$$\frac{U}{U_*} = \frac{1}{k}\log\frac{z}{z_0} \tag{1}$$

where U is the velocity at a height z from the boundary; U_* is defined by $\tau_0 \equiv \rho U_*^2$ where τ_0 is the shearing stress on the boundary and ρ is the density of the fluid; k, von Karman's constant, is about 0·4. For flow over a rough boundary z_0, the roughness parameter, depends only on the geometry of the surface. Over a smooth boundary the effective z_0 is proportional to ν/U_* where ν is the kinematic viscosity.

This expression, with the same value for von Karman's constant, has also been shown to represent the vertical profile of the mean wind near the ground in conditions of neutral hydrostatic stability, so that it seemed likely that it would also represent the flow near the bottom in a tidal stream. Logarithmic profiles have been described by Lesser (1951); the present observations were made in the area where Bowden & Fairbairn had already made independent observations of the shearing stress, in the hope of confirming that the laboratory value of von Karman's constant was again applicable.

There are advantages of working in shallow-water tidal currents; they provide an almost neutrally stable flow of high Reynolds number in which the shearing

stress varies almost linearly from the bottom to the top. There is also a relatively long period variation of the mean velocity with time. Against this one must set the disadvantages of working from a ship; robust instruments are necessary and when in use neither they nor the lower boundary can be inspected.

The results have practical applications not only to tidal dynamics but to the increasing number of problems involving the off-shore disposal of radioactive and other waste.

2. Apparatus

An instrument for measuring the current at various distances from the bottom was constructed and its general arrangement can be seen in Figure 1. It consists

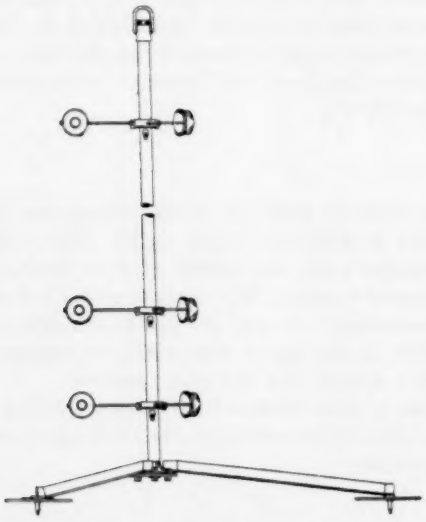

FIG. 1.—Outline drawing of current measuring stand. The vertical tube is 3·8 cm in diameter.

of up to five cupwheels centrally mounted at heights up to 2 m on a vertical steel tube which stands on a tripod base. The cupwheels each have three aluminium alloy cups of almost conical shape and with beaded edges. They rotate on simple ball bearings in Tufnol housings. Stainless steel balls and Monel races resist the seawater corrosion fairly well.

Each cupwheel carries two small magnets by which a simple waterproof switch (see Figure 2) is closed twice per revolution. The magnetic switches are connected by electric cable to a bank of telephone message registers on board the anchored ship.

Some difficulty was experienced initially in preventing the electric cable and the lowering wire from fouling the cups when the ship rolled. The method finally adopted was to lower the stand on 4 mm diameter galvanised steel wire rope to which was attached a small pellet or trawl float a few feet above the top of the stand. The float supports the lowering wire above the cupwheels. The electrical

cable was led from a junction box along one of the legs of the tripod and brought to the ship in a large bight.

The instrument was calibrated on a whirling arm at the Admiralty Research Laboratory, Teddington and in the water channel in the Fluid Dynamics Department of Liverpool University.

3. Observations

Preliminary observations were made from the m.v. *William Herdman* in 1955 July. The site was Red Wharf Bay, an area off Anglesea where Bowden & Fairbairn (1952) had made estimates of the drag coefficient of the seabed. A full description of the area can be found in their paper. Supporting observations were made in the Menai Strait which was used as a shelter from bad weather.

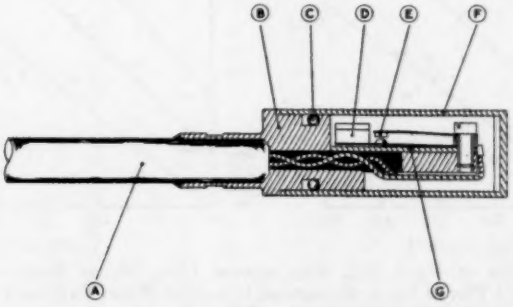

FIG. 2.—Outline drawing of magnetic switch.
A, two-core cable sealed into brass plug B; C, O-ring seal; D, soft iron block; E, silver contacts; F, tubular brass casing, 1·27 cm in diameter; G, insulating pad.

The uppermost cupwheel of the four in use was lost at an early stage so that most of the profiles were based on only three points of which the lowest did not give consistent results. The observations are shown in Figures 3a and 3b in which the mean (10 min average) velocities at 199, 62 and 31·5 cm are plotted against the velocity at 100 cm. There is an indication that the velocities are linearly related and their ratios are plotted against the logarithm of the height in Figure 3c. Apart from the bottom point, which has been ignored in drawing the line, the profile is roughly logarithmic and the roughness parameter appears to be about 0·1 cm. If k in Equation 1 is taken as 0·4 then the corresponding value of the drag coefficient based on the 100 cm current ($C_{100} \equiv (U_*/U_{100})^2$), is $3{\cdot}4 \times 10^{-3}$.

In view of the instrumental deficiencies it was necessary to repeat the observations and an opportunity arose in 1957 July. The measurements were made in the Red Wharf Bay area, again from the m.v. *William Herdman*. On this occasion, thanks to improved technique, all five cupwheels in use survived and individual profiles could be drawn. These are shown in Figure 4 in which the mean velocity is plotted against the logarithm of the height above the bottom. Most of the profiles are approximately logarithmic; the occasional discrepancies could possibly be explained as due to turbulent fluctuations but are more likely to be due to jellyfish which were regrettably numerous on this occasion. The lines drawn on Figure 4

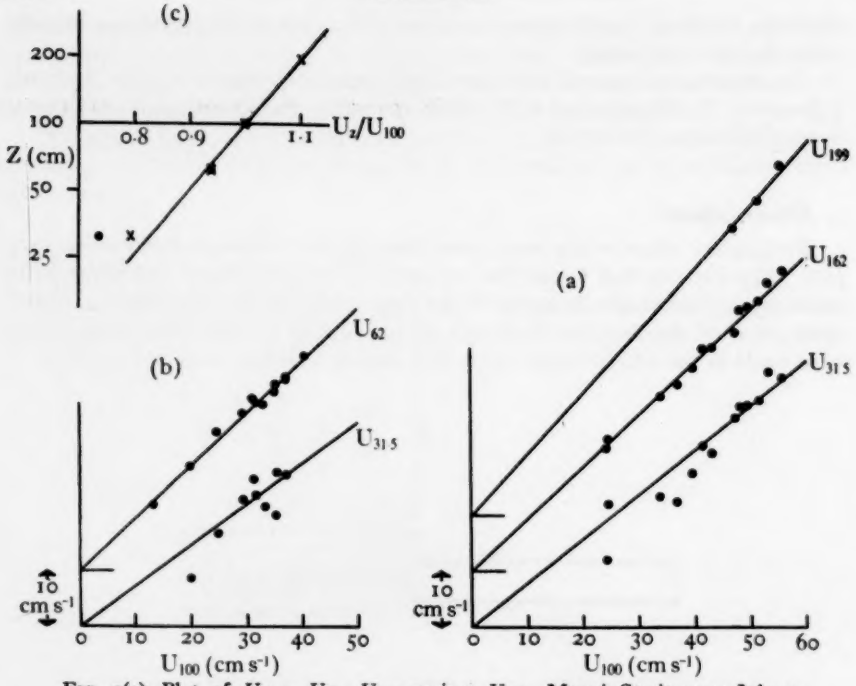

FIG. 3(a) Plot of $U_{31\cdot5}$, U_{62}, U_{199} against U_{100}, Menai Strait 1955 July 25.
(b) Plot of $U_{31\cdot5}$, U_{62} against U_{100}, Red Wharf Bay, 1955 July 26.
(c) Plot of $[U_z/U_{100}]$ against log z.
×—Menai Strait observations. ●—Red Wharf Bay observations.

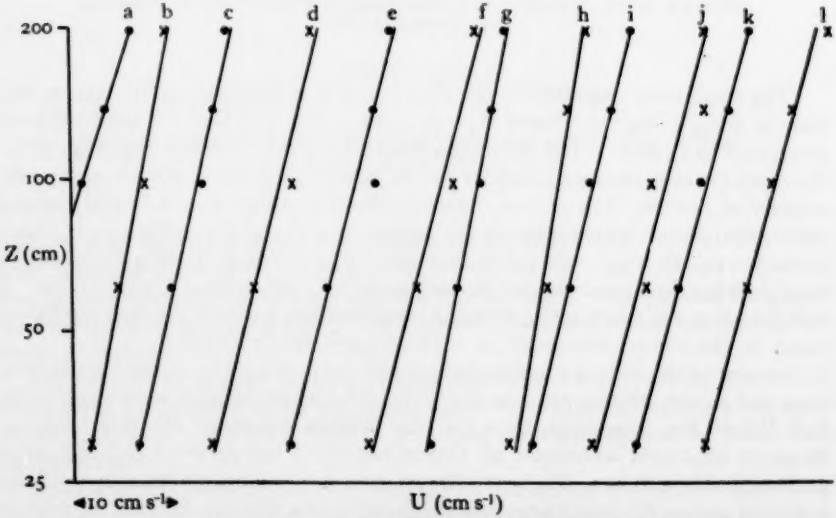

FIG. 4.—Current profiles, Red Wharf Bay, 1957 July 5.
The profiles a, b . . . l were observed at successive times. They are plotted on the same scale but relative to arbitrary origins, the values of U_{200} being given in Table 1.

Table 1

Current profiles observed in Red Wharf Bay, 1957 July 5

Profile	a	b	c	d	e	f	g	h	i	j	k	l
Time GMT	1820	1830	1840	1850	1900	1910	1920	1950	2000	2010	2020	2030
U_{200}(cm/s)	31·2	29·8	31·0	34·6	37·3	40·8	33·4	25·9	35·6	28·1	37·3	38·9
U_*(cm/s)	2·8	1·7	2·0	2·2	2·3	2·3	1·8	1·5	2·2	2·4	2·5	2·4
z_0(mm)	$24 \cdot 8$	$1 \cdot 4$	$3 \cdot 9$	$3 \cdot 9$	$3 \cdot 4$	$1 \cdot 9$	$1 \cdot 0$	$2 \cdot 2$	$3 \cdot 0$	$18 \cdot 2$	$4 \cdot 5$	$3 \cdot 2$
$10^3 C_{100}$	11·6	3·9	5·2	5·2	4·9	4·1	3·4	4·3	4·8	10·0	5·5	4·9

have been drawn by eye and values of U_*, z_0, etc. deduced from them using $k = 0{\cdot}4$ are given in Table 1. Values computed using the least square technique do not differ significantly. It appears that the roughness parameter was about 0·3 cm and this is confirmed by the profiles of mean velocity ratio illustrated in Figures 5a and b. The corresponding value of the drag coefficient based on the 100 cm current is $C_{100} = 4{\cdot}7 \times 10^{-3}$.

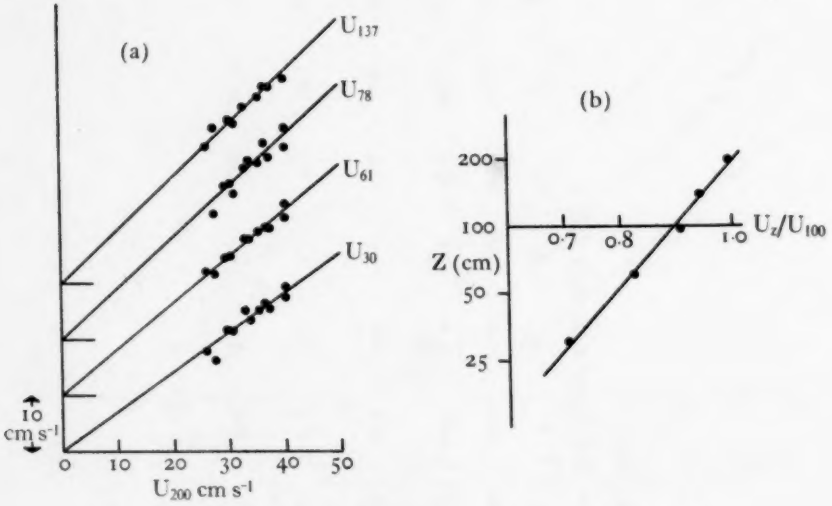

FIG. 5(a) Plot of U_{30}, U_{61}, U_{78}, U_{137} against U_{200}, Red Wharf Bay, 1957 July 5.

(b) Plot of $[U_z/U_{200}]$ against log z, Red Wharf Bay, 1957 July 5.

Discussion

The large amount of long period large-scale turbulence shown by the time variation of U_{200} (Table 1) is noteworthy.

The flow is hydrodynamically rough and the value of 0·1 to 0·3 cm for the roughness parameter ($C_{100} = 3{\cdot}4 \times 10^{-3}$ to $4{\cdot}7 \times 10^{-3}$) is longer than would be expected for smooth sand; it is thought that the bottom was roughened, perhaps by sand ripples.

To compare these results with those of Bowden & Fairbairn (1952) it is necessary to transform their value for the drag coefficient to make it comparable with C_{100}. Their analysis relates the amplitude of the semi-diurnal component of the tidal velocity to the amplitude of the semi-diurnal component of the frictional force; the implied linearization can be regarded as replacing $|\sin \sigma t| \sin \sigma t$ by $(8/3\pi) \sin \sigma t$ and neglecting higher harmonics (see Proudman 1953, p. 309). Their value for the drag coefficient ($1{\cdot}8 \times 10^{-3}$) should therefore be multiplied by $3\pi/8$ to make it comparable with our measurements.

A further adjustment is necessary because their drag coefficient is based on the mean velocity over the whole depth and not on the velocity at 100 cm. They give the mean ratio of the 100 cm and depth mean velocity as 0·77 so that their

estimated value for C_{100} is given by

$$C_{100} = \frac{1{\cdot}8 \times 10^{-3} \times 3\pi}{(0{\cdot}77)^2 \times 8} = 3{\cdot}6 \times 10^{-3}.$$

If all these observations are taken at their face value the value of k was 0·39 on the earlier and 0·46 on the later occasion. It must be remembered that all three sets of observations are subject to observational error, were made at different times and not in precisely the same place, although the bottom is fairly uniform in the area. It seems reasonable to conclude that the laboratory value of von Karman's constant is applicable in tidal streams and that observations of near-bottom currents can provide acceptable estimates of tidal friction.

In a later paper Bowden & Fairbairn (1956) made estimates of the shearing stress on the bottom of Red Wharf Bay from the covariance of vertical and horizontal fluctuations of current. These lead to estimates of drag coefficient which are about half those obtained above. It is possible that some of the momentum flux was missed by the instruments being of insufficiently rapid response or by the observing periods being too short. But a more likely reason for this underestimate is that the corresponding mean currents were not measured directly but were estimated from earlier work. The instrument described in this paper is therefore a useful adjunct to the electrode flowmeter which they have used for their fluctuation measurements; simultaneous observations of mean flow and fluctuations have been made and are now being analysed.

Acknowledgments

The instrument was designed and constructed at the National Institute of Oceanography. The observations in Red Wharf Bay were made from the m.v. *William Herdman* by kind permission of Professor K. F. Bowden.

My thanks are due to the master and crew of the *William Herdman*, to Mr L. A. Fairbairn and to Mr D. I. Gaunt, of the National Institute of Oceanography, who was concerned in the design of the instrument and in the work on board ship.

National Institute of Oceanography,
Wormley,
Godalming,
Surrey.
1959 *May* 5.

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Reports on the Progress of Geophysics

Developments in Dynamical Geodesy*

A. H. Cook

Summary

Progress in dynamical geodesy over the last ten years is reviewed with particular attention to the significance of observations of the orbits of artificial satellites and of direct measurement of the distance of the Moon by radar. As a result of these observations together with a great increase in the number of gravity observations, especially at sea, and with more extensive areas of geodetic triangulation, the ellipticity of the Earth can be determined to one part in 3000 and the radius to a few parts in 100000. It is suggested that the Earth's gravitational field should be specified by the parameter J in the spherical harmonic expansion rather than by the derived ellipticity and that the mean rather than the equatorial values of the radius and sea-level gravity should be employed.

1. Introduction

Dynamical geodesy is the study of the overall shape and size of the Earth and particularly, of the properties of its gravitational field. Its history begins with Newton's study of the motion of the Moon and the shape of the Earth and the problems have engaged many of the most able mathematicians. Until the beginning of this century, theory was far ahead of observation because of the problems of making measurements that were sufficiently exact (relative errors should be less than one part in a million) and were sufficiently numerous and evenly spaced over the Earth. The last few years have seen remarkable changes in this situation, of which some account is given in this review.

The period covered is the ten years or so since Sir Harold Jeffreys published his general adjustment of all the data then existing concerning the shape and size of the Earth (Jeffreys 1948) and to appreciate the significance of the new developments in this period it will be helpful to summarize that paper. The observations which Jeffreys used were:

(a) Measurements of arc lengths and angular amplitudes of triangulation networks—these give the size and ellipticity of the sea-level surface.
(b) Free-air anomalies of gravity.
(c) The luni–solar precession of the Earth.
(d) Certain aspects of the motion of the Moon.
(e) The absolute value of gravity.
(f) The trigonometrical parallax of the Moon.

* *Communication from the National Physical Laboratory*

Except for the Earth's precession, these all involve observations relative to the external gravity field of the Earth and in particular, relative to the equipotential surface which, over the seas, coincides with mean sea-level. This surface will be referred to rather loosely as the geoid. The geometrical form of this surface is related to the external gravity field and, in particular, to the values of gravity on the surface. We are ignoring in this review the details of the local irregularities of the geoid and the gravity field so that these irregularities have the effect of errors in the determination of the gross parameters of the geoid and gravity field.

In the measurements of arcs of triangulations the measurements of the arc lengths can easily be reduced to the geoid since the heights of the observed lines above the geoid can be found from spirit levelling or otherwise. The effect of irregularities in the geoid will then be to make the measured distance slightly greater than that for the best fitting spheroid. This is only a second-order effect. The astronomical observations required for the angular amplitude of the arc are made relative to normals to the geoid and irregularities in the geoid affect the results directly. These irregularities are correlated and Jeffreys discussed in some detail the allowance to be made for them in estimating the uncertainties of the results.

The variation of gravity over an ellipsoidal equipotential surface is related to the geometry of the surface by a well-known theory (Darwin 1899, Jeffreys 1952, Pizetti 1894) which may be extended to cover small irregularities of the surface. The geoid is very closely an exact spheroid but the topographical surface on which gravity observations are made is not an equipotential surface. However, it is known that to a good approximation, the form of the geoid may be found from free-air anomalies. In earlier papers, Jeffreys (1941b, 1943) had studied the spherical harmonic analysis of free-air anomalies and he incorporated the results in his general adjustment. The parameters he used in that adjustment thus depend essentially on observations of gravity prior to 1939 and the distribution of these observations was not very satisfactory for estimating the amplitudes of the various spherical harmonic components since large gaps in the data meant that the estimates of the various coefficients were not independent.

The Moon moves in the Earth's gravitational field and certain periodic perturbations in her motion are due to the second harmonic variation of the Earth's gravity. The quantities are however, very small and the effects depend also on the Sun's gravitational attraction and on the figure of the Moon. In consequence, the standard deviation of the second harmonic in the Earth's field as found from lunar observations is about 5 times that found from surface gravity observations.

The luni–solar precession of the Earth is determined essentially by the ratio $(C-A)/C$ (C is the moment of inertia of the Earth about its polar axis, A that about an axis in the equatorial plane) and the mass of the Moon, although there are also small terms arising from the planets and from general relativity. The coefficient of the second harmonic in the external gravitational potential is proportional to $(C-A)/Ma^2$, where M is the mass of the Earth and a the equatorial radius, whatever the internal constitution of the Earth, but if the stresses within the Earth are exactly hydrostatic, then the ellipticity of the surface may be inferred from the ratio $(C-A)/C$ (Darwin 1899, de Sitter 1915b, Bullard 1948, Jeffreys 1952). At the time when Jeffreys made his analysis, the uncertainty in the value of the ellipticity so derived, due to departures from the hydrostatic state, was comparable with the observational uncertainties of other data from which the ellipticity may be derived.

The absolute value of gravity does not enter the determination of the ellipticity from the variation of gravity with latitude because all the information that can be obtained from that variation is contained in the ratio of gravity at one latitude to that at another and relative measurements of gravity, particularly with pendulums, yield just such ratios. The absolute value of gravity is however required in determining the size of the Earth from the lunar parallax. The value of the Earth's gravitational attraction at the Moon's mean distance R is equal to $R\omega^2$ where ω is the Moon's mean angular velocity. The ratio of this quantity to the mean value of gravity g_m at the surface of the Earth is, by the inverse square law, r_m^2/R^2 where r_m is the mean radius of the Earth.

But the Moon's mean horizontal parallax, $\pi'_{\mathbb{C}}$, is $2r_m/R$ and so

$$r_m = \frac{2\omega^2}{\pi'_{\mathbb{C}}\, g_m}.$$

(These expressions are only approximate, see Section 7.2).

The main uncertainty in applying this equation arises from the measurement of the Moon's parallax. This depends on observations of the Moon at Greenwich and the Cape Observatories and is affected by local deflexions of the vertical at these places as well as by the uncertainty in the distance between the observatories in terms of the Earth's mean radius. The agreement between the lunar parallax as directly observed and as calculated dynamically from survey measurements of the Earth's size has in consequence not been too satisfactory.

By paying careful attention to errors of observation, particularly those which are not independent, Jeffreys was able to derive values of the Earth's equatorial radius and ellipticity which satisfied almost all the data.

His final values were:

$$a = 6378{\cdot}099 \pm 0{\cdot}116\,\text{km},$$
$$\epsilon^{-1} = 297{\cdot}10 \pm 0{\cdot}36,$$

the uncertainties being standard deviations.

The largest change from previous work was the reduction of a by 0·289 km from Hayford's (1910) value, due to the use of free-air instead of isostatic anomalies for the reduction of the astronomical observations.

It is now clear that Jeffreys's values cannot be accepted any longer. This is the consequence of three major technical advances made in the years since 1945. The first of these is the development of methods of measuring gravity rapidly and accurately on land by spring-type gravity meters and at sea by improved submarine pendulum apparatus and more recently by ship-borne gravity meters. Although such methods had given results before 1940, they have been so intensively developed and exploited since 1945 that knowledge of the world-wide features of the Earth's gravity field has increased many-fold.

The second development is the launching of artificial satellites. The motions of these bodies provide estimates of the Earth's ellipticity and other parameters of the external gravity field that are more accurate than Jeffreys's estimates from surface gravity, even though so far only three satellites, and those not with the most favourable orbits, have been observed with the necessary precision.

Finally, it seems that successful radar observations can now be made of the Moon, giving her distance with much greater accuracy than can be obtained from the visual parallax.

In addition the classical method of arc measurements has been extended to much greater areas than heretofore with observations of higher accuracy.

This review is devoted principally to the effects of these major developments but recent work in other fields is also considered. It is first necessary to discuss the theoretical basis of the study of the Earth's gravity field and this is done in the next section, attention being paid particularly to the meaning and definition of the parameters involved.

2. Theory of the Earth's external gravity field

The theory of the Earth's external field comprises the following topics:

(a) Expansion of the potential of an arbitrary body in spherical harmonics. This gives the form of the potential required in the theory of satellite orbits.

(b) Figure of a rotating body with specified moments of inertia bounded by a spheroidal equipotential surface.

This problem admits of an exact treatment and enables the ellipticity of the surface and the variation of gravity over it to be related to the moments of inertia and the external potential.

(c) Distortions of the geoid corresponding to higher harmonics in the potential.

(d) Application of (b) and (c) to the actual Earth which is not bounded by an equipotential surface.

2.1 *Expansion of the potential in spherical harmonics*

The basis of this expansion is MacCullagh's theorem, according to which the first terms of the external gravitational potential V of an arbitrary body are

$$V = \frac{fM}{R} + f\frac{(A+B+C-3I)}{2R^3} + \ldots$$

where f is the Newtonian constant,
M is the mass of the body,
A, B, C are the principal moments of inertia,
R is the distance from the centre of mass to the external point,
I is the moment of inertia about the line joining the centre of mass to the external points.

For the Earth, which is highly symmetrical about the polar axis, C is taken to be the moment of inertia about that axis and A and B to be the (nearly equal) moments about the principal axes in the equatorial plane.

Then if θ is the co-latitude of the external point,

$$I = A\sin^2\theta + C\cos^2\theta$$

and

$$V = \frac{fM}{r_m}\left(\frac{r_m}{R} - \left(\frac{r_m}{R}\right)^3 \frac{C-A}{Mr_m^2} P_2(\cos\theta) + \ldots\right),$$

provided R exceeds the greatest radius vector of the surface of the body.

r_m is the mean radius of the Earth, the definition of which is discussed below. It is usual to express V in terms of (a/R) where a is the equatorial radius but there are good reasons for using r_m instead, as was first indicated by de Sitter (1915a). Most observations relating to the size of the Earth—geodetic measurements of

arcs, visual lunar parallax, radar observations of the Moon—are made in middle latitudes and the quantity which is most directly involved is either the radius of curvature of the arc or the radius vector of the observatory. It is desirable to state the theory in terms of a parameter as nearly equal as possible to the distance concerned and not one, such as the equatorial radius, which is derived from them by a relation involving the ellipticity. Furthermore, the quantity m, equal to the ratio of centrifugal force at the equator to gravitational attraction, is almost independent of e when expressed in terms of mean gravity and mean radius. Formally it does not matter what parameter, a, r_m or some other, is used in MacCullagh's theorem, although a change in the definition of J (that is $\frac{2}{3}(C-A)/Ma^2$) is also involved, and in this review the theorems will be stated in terms of r_m.

2.2 *Figure of a rotating spheroid*

The theory of the figure of a spheroidal equipotential surface bounding a rotating body has been established by two methods. In the first (Darwin 1899, de Graaff Hunter 1935) the external potential has been expanded in a series of spherical harmonics. Is it open to the criticism that spherical harmonics are not orthogonal in the space outside a spheroid but as far as terms of order e^2 the same results are obtained by the second method (Pizetti 1894) in which ellipsoidal functions orthogonal outside an ellipsoid are employed. Alternatively, spheroidal harmonics may be used, as has been done in carrying the theory to include terms of order e^3 (Cook 1959), an extension that seems desirable in view of the latent precision of satellite observations.

The important results are the following; they are stated in terms of the mean radius and mean gravity defined as

$$r_m = a(1-\tfrac{1}{3}e-\tfrac{1}{3}e^2)$$

and

$$g_m = \frac{fM}{r_m^2}\left\{1-\frac{2}{3}m-\frac{2}{9}e\left(\frac{19e}{5}-5m\right)\right\}.$$

These are the mean values of radius vector and gravity expressed as functions of geocentric co-latitude.

Gravity formula in terms of ellipticity:

$$g/g_e = 1+\left(\frac{5}{2}m-e+\frac{15m^2}{4}-\frac{17em}{14}\right)\sin^2\phi+\frac{1}{8}e(e-5m)\sin^2 2\phi;$$

$$g/g_m = 1+\frac{1}{3}e+\frac{5}{6}m+\frac{8}{9}e^2-\frac{41}{14}em-\frac{5}{9}m^2+\left(\frac{5}{2}m-e-\frac{e^2}{3}-\frac{17e}{14}m+\frac{35}{6}m^2\right)\sin^2\phi+$$
$$+\frac{1}{8}e(e-5m)\sin^2 2\phi.$$

Relation between ellipticity and moments of inertia:

$$J = e-\frac{1}{2}m-\frac{1}{2}e^2+\frac{1}{7}em;$$

$$J^* = e-\frac{1}{2}m+\frac{1}{6}e^2-\frac{4}{21}em;$$

$$e = J^*+\frac{1}{2}m-\frac{1}{6}J^{*2}+\frac{5}{56}m^2+\frac{1}{42}J^*m.$$

Gravity formula in terms of moments of inertia:

$$\frac{g}{g_m} = 1 + \frac{1}{3}J^* + m + \frac{5}{6}J^{*2} - \frac{128}{63}J^*m - \frac{99}{56}m^2 + \\ + (2m - J^* - \frac{1}{6}J^{*2} - \frac{11}{7}J^*m + \frac{283}{56}m^2)\sin^2\phi + \frac{1}{8}(J^{*2} - 4J^*m - \frac{9}{4}m^2)\sin^2 2\phi.$$

The symbols are as follows:

ϕ is geographic latitude,

$$J = \frac{3}{2}\frac{C-A}{Ma^2},$$

$$J^* = \frac{3}{2}\frac{C-A}{Mr_m^2},$$

$$m = \frac{\omega^2 r_m^3}{fM},$$

(ω is the angular velocity of the Earth)

g_e is the value of gravity at the equator,

$$\frac{g_e}{g_m} = 1 + \frac{1}{3}e + \frac{5}{6}m + \frac{8}{9}e^2 - \frac{41}{14}em - \frac{5}{9}m^2.$$

2.3 *Application to the actual Earth*

Whereas a satellite moves in the potential field of the actual Earth with an irregular surface, so that from its motion the true moments of inertia of the Earth are obtained, the formula with which surface gravity is compared applies to an idealized body bounded by an equipotential surface. To first order, the actual gravity values may be brought into correspondence with the idealized ones by applying the free-air correction—it has been pointed out that the significant quantity here is the potential difference between the point of observation and the geoid and not the gravity difference (Cook 1953). The more exact theory of this correspondence probably requires to be re-examined especially as a second harmonic in the distribution of land might introduce an error into the estimated second harmonic of free-air anomalies. The usual approach is to set up a model Earth with the correct second-harmonic field but with the higher harmonics giving the correct value of gravity at the external surface only. This avoids a number of difficult questions and is sufficient for a first-order treatment of the departures from the second harmonic field.

3. Spherical harmonic analysis of gravity values

The principal difficulty in the analysis of gravity values is the uneven distribution of observations together with the correlation between gravity values at places separated by very considerable distances.

The net effect is that the coefficient of any one harmonic depends to some extent on all the others whereas if the observations were uniformly distributed all the coefficients would be independent. The situation has greatly improved since

Jeffreys made his analysis (1941b). He then had available fewer than 10 000 observations and almost the whole of the South Pacific and the southern oceans as well as the polar regions were unsurveyed. Nowadays many land areas are covered quite densely with observations made with gravity meters, some seas especially the Mediterranean, are adequately surveyed and the observations in the North Atlantic and the North Pacific oceans have been greatly extended. Observations have also been made in Canada as far north as latitude 82°30′. The main gaps now lie in the southern oceans and Australia; in addition, Russian results are not generally available*. Very many observations indeed have been made in some land areas and the problems of handling all the data and reducing them to forms suitable for geodetic use are very considerable. One would like to have mean free-air anomalies in areas one degree of latitude by one degree of longitude, together with the mean heights of these areas, for it is necessary to analyse the variation of free-air anomaly with height. Two groups are working on this problem—the International Gravity Bureau in Paris which, until his untimely death was under the direction of Dr Lejay, and Professor W. A. Heiskanen's group at Columbus, Ohio. Few results have so far been published although Professor Heiskanen has summarized his work (Heiskanen 1957). Even now, only about 15 per cent of all one-degree squares have observations.

The effect of correlation and the uneven distribution of observations was thoroughly discussed by Jeffreys (1941b, 1943) while Kaula (1957) has developed an alternative approach which may be applied to the problem of spherical harmonic analysis. The results of analyses are expressed in the form

$$g = g_e(1 + \beta \sin^2\phi - 5{\cdot}9 \times 10^{-6} \sin^2 2\phi).$$

The term with $\sin^2 2\phi$ is calculated theoretically from the condition that the equipotential surfaces are exact spheroids. g_e and β are determined from the observations. Some results are given in Table 1.

Table 1

Results of spherical harmonic analysis of free-air anomalies

Reference	g_e	β	e^{-1}	J
Jeffreys, 1948	978·051cm/s^2	$5{\cdot}2863 \times 10^{-3}$ $\pm 0{\cdot}0075$	296·85 $\pm 0{\cdot}66$	$1{\cdot}640 \times 10^{-3}$
Heiskanen, 1957	978·0497	5·2902	297·2	1·633
Krassovski, 1940	978·049	5·3029	298·4	1·622
Zhongolovitch, 1952	978·0573	5·2837 $\pm 0{\cdot}0064$	296·61 $\pm 0{\cdot}57$	1·642

Heiskanen's result has been criticized by Jeffreys (1959) for neglect of the effects of correlation.

There are two theoretical points in the analysis of gravity observations that require study. One is the effect of low order harmonics in the height of the topography which may come from two causes, namely from any error in the free-air correction formula and also, through the imperfect formation of means, from the

* But see Zhongolovitch 1952.

correlation of free-air anomaly with height. The second point is that anomalies are always given in terms of geographic latitude. However, since they are referred to a surface which is a spheroid they should be analysed in terms of functions that are orthogonal on a spheroid—that is, Legendre functions of confocal coordinates (see Cook 1959).

4. Motion of an artificial satellite

4.1 *Theory of perturbations*

Within recent years a great deal of work has been done on the motion of a satellite in the Earth's gravity field. Some of the earlier studies are too restricted or are in error and present determinations of e and J from satellite orbits depend on the work of King-Hele (1958), of Herget (1957) and of Krause (1956).

King-Hele has carried the theory as far as terms of the order of J^2. He takes the gravitational potential of the Earth in the form

$$\frac{fM}{a}\{a/r-(a/r)^3\frac{2}{3}J P_2(\cos\theta)-(a/r)^5\frac{8}{35}D P_4(\cos\theta)\},$$

where r is the radius vector and θ the co-latitude of the satellite, and a is the Earth's equatorial radius. He then finds by a perturbation method that the rate at which the plane of the orbit rotates about the polar axis of the Earth is given by:

$$\frac{d\Omega}{dt}=\left(\frac{fM}{R^3}\right)^{\frac{1}{2}}(a/l)^2\cos\alpha\times$$
$$\times[J+J^2(a/l)^2\left(\frac{19}{12}\sin^2\alpha-1\right)-(3D/14)(a/l)^2(7\sin^2\alpha-4)+O(J^2e)];$$

the rotation is in the opposite sense to the motion of the satellite in its orbit. Here R is the semi-major axis of the orbit, l the semi-latus rectum, e the eccentricity of the orbit and α the mean inclination of the orbital plane to the equator.

The other principal secular effect is the rotation of the major axis which King-Hele finds to be given by

$$\frac{d\beta}{dt}=J(\gamma/\alpha)^{\frac{1}{2}}(a/l)^2(a/R)^{\frac{1}{2}}\tfrac{1}{2}(5\cos^2\alpha-1)+O(Je)+O(J^2)$$

where $\gamma=fM/a^2$.

When $\cos^2\alpha=1/5$, the rate of rotation is zero; for equatorial orbits, the axis rotates in the same direction as the satellite and for polar orbits in the opposite direction.

King-Hele considered just the second and fourth harmonic terms in the Earth's potential but results of the spherical harmonic analysis of free-air anomalies do not exclude the existence of other harmonics.

R. H. Merson and R. N. A. Plimmer of the Royal Aircraft Establishment are extending the theory, by a method somewhat different from King-Hele's, to take account of other harmonics and find, for instance, that if there is a third harmonic with coefficient J_3, it gives rise to a component of the rotation equal to

$$-\tfrac{3}{8}(fM/R^3)^{\frac{1}{2}}(a/l)^3 J_3 e\sin\omega\cot\alpha(15\sin^2\alpha-4).$$

(Merson & King-Hele 1958). ω is the mean argument of perigee. This rotation is

evidently very small for nearly circular orbits so that J_3 can only be well determined from highly elliptical orbits. J. de Graaff-Hunter is also studying orbital theory. American workers have used the theory developed by Krause (1956) and that of Herget (1957). Krause neglects the J^2 term yet keeps the D term which is of the same order.

Superposed on the effects of the Earth's gravitational field are those due to the resistance of the air. The main effects of the atmosphere are to reduce the major axis, to reduce the eccentricity of the orbit and to reduce the period. These are quite different from the effects of the oblateness of the Earth and are readily separated from it by observation. The theory of the effects of air drag has been studied by many authors and so far there is no indication that drag will cause rotation of the plane of the orbit or of the major axis. A lateral force on the satellite due to rotation of the atmosphere might produce a rotation of the orbital plane and this would be revealed most directly by an orbit passing over the poles; another way of testing for such an effect would be to have satellites going westwards as well as eastwards round the Earth. (King-Hele & Merson 1958).

So far the attractions of the Sun and Moon have been ignored; they cause perturbations amounting to about 1 in 10 000 of those due to the oblateness of the Earth and it may eventually be necessary to allow for them, particularly if orbits with appreciably greater axes are used. There is also a very small effect from general relativity.

4.2 *Reduction of satellite observations*

A term in the gravitational potential that is periodic in longitude will not produce any secular effect on the satellite orbit unless the period happens to coincide with that of the satellite. Such effects are ignored at present and we consider only the perturbations due to terms in the potential proportional to the Legendre functions $P_2(\cos\theta)$, $P_3(\cos\theta)\ldots$ with coefficients J_2, J_3, ...*. Then in general, the orbital plane will rotate at a rate

$$C_2J_2+C_3J_3+C_4J_4+\ldots$$

while the major axis will rotate at a rate

$$d_2J_2+d_4J_4+\ldots$$

where the coefficients $C_2\ldots d_2\ldots$ vary with the parameters of the orbit and in particular, C_4 vanishes for $\sin^2\alpha = 4/7$ and d_2 for $\cos^2\alpha = 1/5$.

In favourable circumstances observations of both quantities for one satellite might enable J_2 and J_4 to be found separately. More generally, it should be possible to determine a number of the J's by combining the observations on different satellites. At present it is easier to get accurate observations of the rate of rotation of the orbital plane than that of the major axis, since the plane is the better defined.

The Russian satellite, Sputnik 2 (1957β), has been observed optically by kinetheodolite (Merson & King-Hele 1958) and the rate of rotation of the orbit is quite well defined. The method of analysis is described by King-Hele & Merson (1958). The result can be expressed as follows:

$$1{\cdot}0008j-0{\cdot}312d = (-6{\cdot}8\pm0{\cdot}3)\times10^{-3}$$

where $j = \delta J/J$, $d = \delta D/J$ and δJ and δD are the differences from Jeffreys's values $J = 1{\cdot}637\times10^{-3}$ and $D = 10{\cdot}6\times10^{-6}$.

* Note that $J = -\frac{1}{2}J_2$, and $D = \frac{1}{35}J_4$ in the conventional expression for the potential.

Sputnik 2 was also observed visually and photographically at eleven observatories in Czechoslovakia, mostly by plotting the position on a star chart. Buchar (1958) has determined the motion of the ascending node from these data and has calculated J from the result using a formula which agrees with King-Hele's as to the terms proportional to J and D but from which the J^2 term has been omitted. In his numerical work, Buchar assumes that D has the theoretical value for a perfect spheroid, namely $10{\cdot}7\times10^{-6}$, and then gets

$$J = (1{\cdot}6278 \pm 0{\cdot}0020)\times 10^{-3}$$

corresponding to $e^{-1} = 297{\cdot}9 \pm 0{\cdot}18$.

The rate of rotation of perigee (major axis) of the orbit of Sputnik 2 is very small because $5\cos^2\alpha$ is close to 1. The American satellites have a larger rate of rotation of perigee and J and D can be found separately by combining results from observations of this motion with observations of the motion of the node (rotation of the orbital plane). American satellites are observed by the Minitrack stations. Harvard announcement card 1408 of 1958 June 14 gives the following values of e^{-1}.

Satellite	Observation	e^{-1}
1958α	Node	$298{\cdot}0 \pm 0{\cdot}3$
	Perigee	$297{\cdot}8$
1958β	Node	$298{\cdot}38 \pm 0{\cdot}07$
	Perigee	$298{\cdot}3$

These values appear to be derived on the assumptions that J^2 is negligible and the D has the value $10{\cdot}6\times10^{-6}$ for a truly spheroidal Earth.

Cook (1958) has combined King-Hele and Merson's result with Jeffreys's (1943) values of the coefficients of the second and fourth harmonics in gravity, obtaining

$$J = (1{\cdot}628 \pm 0{\cdot}004)\times 10^{-3}$$
$$D = (15{\cdot}7 \pm 8{\cdot}2)\times 10^{-6}*$$

This is not really a satisfactory procedure since it combines two sets of data that one would like to compare separately and better results are obtained by combining the results for Sputnik 2 and the American satellites. In this way Merson finds the following equations:

From Sputnik 2

$$J - 0{\cdot}311D = 1{\cdot}6227\times 10^{-3}$$

from American satellites

$$J + 0{\cdot}2645D = 1{\cdot}6264\times 10^{-3}$$
$$J + 0{\cdot}2035D = 1{\cdot}6278\times 10^{-3};$$

the least squares solution is

$$J = (1{\cdot}625 \pm 0{\cdot}001)\times 10^{-3}$$
$$D = (8 \pm 3)\times 10^{-6}$$
$$e^{-1} = 298{\cdot}2 \pm 0{\cdot}1.$$

* There is a misprint in the original paper.

(Geophysical Discussion of the Royal Astronomical Society, 1959 January 23—see *Geophys. J.* **2** (1959) p. 251).

Better equations for Vanguard 1 (Jacchia 1958) and Explorer 4 are now available and the definitive results are those of King-Hele & Merson (1959) who have combined observations of the rates of rotation of the orbital planes of Sputnik 2 (1957β), Vanguard 1 (1958β2) and Explorer 4 (1958ϵ) and find:

$$10^6 J = 1624{\cdot}6 \pm 0{\cdot}3, \quad 10^6 D = 6 \pm 1,$$
$$10^6 J_6 = 0{\cdot}1 \pm 1{\cdot}5 \quad \text{and} \quad e^{-1} = 298{\cdot}20 \pm 0{\cdot}03.$$

To summarize the results so far obtained, the value of J seems to be determined at about $1{\cdot}625 \times 10^{-3}$, D appears to be less than its conventional spheroidal value of $10{\cdot}6 \times 10^{-6}$ and J_6 is scarcely significant.

In addition, observations on the American satellite Vanguard 1 suggest that the term proportional to $P_3(\cos\theta)$ in surface gravity has an amplitude of about 5 mgal (O'Keefe, Eckels & Squires 1959).

It is desirable that the results of satellite observations should be given primarily as coefficients of spherical harmonics in the potential rather than as the corresponding ellipticity because although the latter is clearly defined as $(a-b)/a$ for a true spheroid, the definition can be ambiguous if the equipotential surfaces depart from spheroids. The coefficients of the spherical harmonics on the other hand, are unambiguous physical properties of the Earth and are the quantities that directly determine the motion of a satellite.

4.3 *Comparison with surface gravity observations*

Jeffreys's analysis of free-air gravity anomalies gives

$$10^3 J = 1{\cdot}640 \pm 0{\cdot}007; \qquad e^{-1} = 296{\cdot}85 \pm 0{\cdot}66$$

while the satellite observations give

$$10^3 J = 1{\cdot}6246 \pm 0{\cdot}0003; \qquad e^{-1} = 298{\cdot}2 \pm 0{\cdot}03.$$

Jeffreys (1959) has discussed possible reasons for this discrepancy but in view of the fact that smaller values for e have been obtained from the same data (see Section 3) it is not unlikely that, when the more recent gravity observations have been incorporated into the analysis, better agreement will be obtained. A discussion of the available material on the lines of Jeffreys's analysis is very desirable.

5. The moments of inertia of the Earth

The effect of the oblateness of the Earth is that the attraction of the Sun exerts a couple tending to rotate the Earth about an axis in the plane of its orbit and perpendicular to the line joining the Sun and the Earth and in consequence. the Earth precesses at a mean rate equal to

$$\frac{3}{2}\frac{n'^2}{\omega}\frac{C-A}{C}\cos\theta,$$

where n' is the Earth's angular velocity in its orbit,
ω is the Earth's angular velocity about its polar axis
and θ is the angle between the Earth's polar axis and the pole of the ecliptic.

The Moon also contributes to the precessional couple and gives a precession equal to

$$\frac{3}{2}\frac{C-A}{C}\frac{n''}{\omega}\frac{1}{1+\mu}\cos\theta',$$

where n'' is the Moon's angular velocity in her orbit,

μ is the ratio of the mass of the Earth to the mass of the Moon,

θ' is the angle between the Earth's polar axis and the pole of the Moon's orbit.

The total luni–solar precession is about 50″ per year of which about 15″ is due to the Sun.

There are some minor complications in the theory. In the first place, allowance has to be made for the effect of the other planets. There is also a small correction entailed by general relativity. The net precession due to the Sun and Moon is 50″·39 per annum.

The value of μ is required in order to calculate $(C-A)/C$ denoted by H and this, and the constant of precession P have been redetermined by Rabe (1950a, b) from observations of Eros as follows:

$$P = 5493''\cdot793$$
$$\mu^{-1} = 81\cdot375;$$

these lead to

$$H = 3\cdot27559\times10^{-3}.$$

Jeffreys's (1948) value was $3\cdot27237\times10^{-3}$.

If it is assumed that the stresses in the Earth are hydrostatic, then J may be calculated from H (Jeffreys 1952). Since H is exceedingly well known, the uncertainty in the value of J found in this way comes entirely from ignorance of the true state of stress in the Earth. Jeffreys (1948) estimated the uncertainty of J to be 3×10^{-6} which makes this estimate of J less uncertain than those from any data other than satellite observation. The satellite value is however more precise so that it may be used in conjunction with H to compute the moments of inertia.

We have

$$H = 0\cdot0032756;$$
$$J = 0\cdot001624^{6},$$

or

$$\frac{C-A}{Ma^2} = 0\cdot0010831.$$

Hence

$$\frac{C}{Ma^2} = \tfrac{2}{3}(J/H) = 0\cdot3306,$$

and

$$\frac{A}{Ma^2} = 0\cdot3295.$$

If the Earth were in a hydrostatic state, e^{-1} would be 297·10, the corresponding value of J being $1\cdot637\times10^{-3}$. This value of e^{-1} is greater than Bullard's (1948) value because of the reduction in H.

The comparison of the satellite value of J with that computed from the luni–solar precession demonstrates that the stresses in the Earth are not hydrostatic,

as is of course evident from the existence of mountains and ocean basins as well as from the large-scale departures from isostacy.

6. The motion of the Moon

Just as with an artificial satellite, the plane and the major axis of the Moon's orbit rotate on account of the oblateness of the Earth. The theory was given in its most complete form by E. W. Brown (1908). At the distance of the Moon the higher harmonics have no effect but the rates of rotation depend also on the oblateness of the Moon and further, are much smaller than the rates of rotation due to the Sun and planets. Thus it is difficult to obtain accurate estimates of the parts that depend on the figures of the Earth and Moon while the solution for the Earth's oblateness depends on that of the Moon. Jeffreys (1937, 1941a, 1948) discussed these problems and used lunar data in his general adjustment but now that the Earth's oblateness is so well determined by satellite observations, it is possible to substitute the value of J in the lunar equations and re-examine the values of the lunar parameters. In addition to the quantity J the following parameters referring to the Moon, are required:

$$J' = \frac{3}{2}\frac{2C'-A'-B'}{2M'a'^2}, \qquad K' = \frac{3}{2}\frac{B'-A'}{M'a'^2},$$

$$\beta = \frac{C'-A'}{C'}, \qquad \alpha = \frac{C'-B'}{C'}, \qquad q' = \frac{3}{2}\frac{C'}{M'a'^2}, \qquad L' = J' + \tfrac{1}{2}K'.$$

C' is the moment of inertia about the axis of rotation, A' that about the axis pointing to the Earth and B' about that tangential to the orbit. a' is the equatorial radius.

The following equations are then found for the motions of the node (rotation of plane) and of perigee (rotation of major axis):

Perigee $3897J + 380L' - 1192K' = 6''{\cdot}411 \pm 0{\cdot}039$ per year;
Node $-3648J - 460L' = -6''{\cdot}179 \pm 0{\cdot}039$ per year.

(the motions due to solar perturbations are $146427''$ for the perigee and $-69672''$ for the node).

Taking J to be $1{\cdot}625 \times 10^{-3}$,

$$L' = 5{\cdot}46 \times 10^{-4},$$

$$K' = 1{\cdot}28 \times 10^{-4},$$

and so

$$J' = 4{\cdot}82 \times 10^{-4}.$$

The value of β is related to the inclination of the Moon's axis and de Sitter (1915c) gives

$$\beta(1 + 0{\cdot}0047f) = 0{\cdot}0006286 \pm 0{\cdot}0000015.$$

f is equal to α/β, that is to $1 - K'/L'$.

f may be determined from observations of the Moon's libration in longitude though until recently the results were very discrepant. Recent discussions of new

observations appear to give more concordant results and Jeffreys (1957) finds

$$\beta = 0{\cdot}0006266 \pm 0{\cdot}000\,0022$$

and

$$f = 0{\cdot}673 \pm 0{\cdot}0018,$$

while Habibulin (1958) obtains

$$f = 0{\cdot}60 \pm 0{\cdot}03$$

which entails

$$\beta = 0{\cdot}000\,6279.$$

Now the above values of K' and L' lead to $f = 0{\cdot}77$ which appears to be significantly greater than either Jeffreys's or Habibulin's value.

There is a further difficulty. We have

$$\frac{3}{2}\frac{C'-A'}{M'a'^2} = L'$$

and thus

$$q' = \frac{3}{2}\frac{C'}{M'a'^2} = L'/\beta,$$

which with the above numerical values gives

$$q' = 0{\cdot}87.$$

This however, implies that the interior of the Moon is considerably less dense than the outer parts (if the density were uniform, q' would be 0·6). It seems therefore, that there are inconsistencies in lunar observations which are not removed by an improved value of J.

7. The size of the Earth

7.1 *The radius from arcs of geodetic triangulation*

The radius of the Earth has for many centuries been found from measurements of the distances (arc-lengths) between places of which the geographical latitudes are known from astronomical observations. Among such measurements of historical importance are the French measurements of arcs at the equator and in Lapland which showed that the Earth was, as Newton maintained, an oblate spheroid, and the measurements of the arc through France and Spain from which the ratio of the Metre to the Toise was obtained. If the observations were made on the surface of a spheroid the errors of the calculated radius would be determined solely by the errors of observation. In practice, the observations are made at places which may be some distance above sea-level and are referred to surfaces—the equipotential surfaces passing through the points of observation—which are not true spheroids. The height above sea-level can be measured by spirit-level observations and corrections can be applied to reduce the observations to sea-level but the irregularities of the equipotential surfaces cause more trouble. If gravity surveys extended over the whole surface of the Earth, the deflexions of the vertical could be calculated at all points along an arc and the corresponding geographical latitudes on a spheroid could be calculated from the true observed latitudes. At present, however, this is not possible and the most serious difficulties arise not from those parts of the deflexions due to local mass irregularities

but from those which maintain the same sign over a large part of an arc of observation. The reduction of the observations taking these correlated errors into account was discussed by Jeffreys (1948). A further difficulty is that the observed arcs do not extend from pole to pole and so instead of giving a and e separately, they are best considered as giving a relation between a and e. Then if e is found from other data, a may be calculated. In this way, Jeffreys obtained:

$$a = 6378{\cdot}099 \pm 0{\cdot}116\,\text{km}$$

for

$$e^{-1} = 297{\cdot}10.$$

In this result some weight was given to an equation from the Moon's parallax (see Section 7). Jeffreys's result differs from the value due to Hayford (1910) adopted for the International Spheroid largely because Hayford applied isostatic corrections to the observed latitudes, a process which may introduce systematic errors. Jeffreys's discussion of the observations available up to 1948 is the most thorough; the results up to 1950 have been summarized by Strasser (1957).

These earlier results have been superseded with the completion of two major arcs of geodetic triangulation, that in North and South America and that from Scandinavia to South Africa where observations on the arc of the 30th meridian have at last been finished. Using these new data, Chovitz and Fischer (1956) have made two solutions; one with isostatic reductions and the other without. They assume that $e^{-1} = 297 \pm 1$ and obtain

$$a = 6378{\cdot}240\,\text{km} \pm 100\ \text{(free–air)}$$

and

$$a = 6378{\cdot}285\,\text{km} \pm 100\ \text{(isostatic)}.$$

The uncertainties arise almost entirely from the uncertainty of e^{-1}. If e^{-1} is taken to be $298{\cdot}2 \pm 0{\cdot}03$ (Section 4.2) the values become

$$a = 6378{\cdot}126\,\text{km} \pm 50\ \text{(free–air)},$$

$$a = 6378{\cdot}165\,\text{km} \pm 18\ \text{(isostatic)}.$$

The reduction in the uncertainties is the consequence of the higher precision of e^{-1}.

The difference of the free-air value for $e^{-1} = 297$ from Jeffreys's value is due to the much longer arcs that Chovitz & Fischer used.

Mrs Fischer (1959) has subsequently made a new estimate of a from observations of the height of the geoid above the reference spheroid. The material comes from large areas in N. America, Europe and the USSR as well as from smaller regions in Africa and S. America. The data are effectively of the same kind as measurements along arcs of triangulation but cover areas rather than lines and the results correspond to the "free-air" values above. For the conventional value $e^{-1} = 297$ (exact)

$$a = 6378{\cdot}253\,\text{km}$$

and with the satellite value $e^{-1} = 298{\cdot}2 \pm 0{\cdot}03$, a will be somewhat less, about 6378·200 km. No firm estimate of the uncertainty has been given.

7.2 *The radius from the distance of the Moon*

The distance of the Moon is connected with the radius of the Earth as follows:

Apart from perturbations due to the Sun and other causes, the Moon's orbit

about the Earth is an ellipse of which the semi-major axis $\bar{R}$ is given by

$$n''^2\bar{R}^3 = fM$$

where n'' is the Moon's mean angular velocity in her orbit and M is the mass of the Earth. Now the product fM is obtained from gravity observations on the surface of the Earth; in fact for a spheroid;

$$g_m = \frac{fM}{r_m^2}\left\{1 - \frac{2}{3}m - \frac{2e}{9}\left(\frac{19e}{5} - 5m\right)\right\}.$$

(The value of gravity should be increased by about 1 in 10^6 for the attraction of the atmosphere which does not enter surface measurements.) Thus the observed value of $\bar{R}$ when combined with the very accurately known value of n'' and with the value of g_m, gives an equation for r_m which is almost independent of e.

Until very recently, $\bar{R}$ had to be found trigonometrically from the Moon's parallax $\pi\mathbb{1}$. The definitive geometrical value is derived from observations of Christie & Gill (1911) at the Greenwich and Cape Observatories. However, the distance between these places had not then been measured directly and so had to be calculated from the geographical latitudes using assumed values of a and e. In addition there may be a systematic error in the parallax due to the deflexions of the vertical at the observatories. O'Keefe (1950) in the United States, has effected a much more satisfactory trigonometrical determination from photoelectric observations of occultations made at places the relative positions of which could be found directly from the geodetic triangulation of the United States without any assumption of values of a and e. He finds

$$\bar{R} = 384410\,\text{km}.$$

More recently the distance of the Moon has been measured successfully by radar at the Naval Research Laboratory, Washington D.C. (Yaplee, Bruton, Craig, & Roman 1958) and at the Royal Radar Establishment, Malvern (Hey & Hughes 1958).

Although the results are only preliminary they show that it should be possible to obtain $\bar{R}$ to almost one part in a million, provided that the theory of the Moon's motion is adequate. Brown (1905) considers that his calculated values of the lunar parallax are correct to $0''{\cdot}01$, that is, to about 3 in 10^6, so that the present theory may not be quite good enough. This depends to some extent on how the observations are made. The effects of the Sun on the Moon's orbit are to increase its size and to introduce periodic perturbations with arguments such as the difference of the longitudes of the Sun and the Moon. If the observations were to extend over a sufficiently long time, the periodic perturbations could be determined observationally but if the observations cover only a short time interval, then the coefficients of the periodic terms must be calculated theoretically accurately enough to apply corrections for them. In any case, the constant term should be evaluated theoretically to about one part in a million of the radius vector.

Radar observations and occultation observations actually determine the distance D_{obs} between the observatory on the Earth and a point on the surface of the Moon. Yaplee and others and Hey and Hughes both find that the effective reflecting surface of the Moon for 10 cm radio-waves is a fairly small area normal to the line joining the observatory and the centre of the Moon. If the length of the radius vector of this part of the surface of the Moon is assumed to be known, then the

equivalent distance D from the observatory to the centre of the Moon may be found. Now let r be the length of the radius vector of the observatory and θ the angle between it and the line joining the observatory to the centre of the Moon. Hence

$$D^2 = R^2 + r^2 - 2Rr\cos\theta$$

where R is the distance of the Moon at the time of observation.

R can be calculated in terms of $\bar{R}$ and r in terms of r_m and e. If $\bar{R}$ is replaced by its expression in terms of g_m, r_m, e, and m, we have an equation for D involving these quantities (the non-spherical shape of the Moon should also be allowed for). D can be determined to about 1 in 300 000 for a single value of θ and if the observations are extended over a long period the resulting combined equation should have a weight rather higher than that of equations derived from surface measurements on the Earth. In this equation m has an uncertainty of 6×10^{-8}. g_m depends on measurements of the absolute value of gravity as well as on relative measurements; measurements now in progress (see Cook 1957) should give the former to 1 in 10^6 while the latter depends on e. Moon radar observations should therefore give an equation between r_m and e with relative errors of the order of 1 in 10^6 as compared with the equations from measurements over the surface of the Earth which can scarcely have greater precision than 1 in 10^5.

Moon radar observations for this purpose have to be made at a wavelength of about 10 cm. The available power decreases at shorter wavelengths while at much longer wavelengths the echoes fade in a complicated way and the refractive index of the ionosphere introduces large errors. The difficulties do not occur at 10 cm. In the two experiments so far made the pulses have been about 5 μs long and a resolution of about $\frac{1}{2}$ km has been achieved.

Summary

Since Jeffreys's 1948 discussion, quite new ways of finding a and e have been developed and in addition, much new material for the classical methods of gravity survey and arc measurements has been acquired. The new gravity material has not been fully discussed so far and radar observations of the Moon have not given final results but very good values of J and e come from satellite observations and of a from arc measurements and geoidal heights. These are

$$10^3 J = 1{\cdot}6246 \pm 0{\cdot}0003$$

$$e^{-1} = 298{\cdot}20 \pm 0{\cdot}03$$

$$a = 6378{\cdot}200\,\text{km}.$$

The uncertainties in r_m and g_m are of the order of e^2(1 in 10^8) and if with further improvements in technique, uncertainties of the order of 1 in 10^6 can be achieved, questions of the definition of parameters such as g_m, r_m and e will need to be re-examined. Satellite observations, it must be emphasized, give physical parameters such as J whereas the reduction of all other observations involves in some way the use of a best fitting spheroid which has no physical existence.

The launching of artificial satellites is stimulating new work on the overall physical properties of the Earth and, in turn, improved values of the astronomical constants of the Earth, Moon and Sun will be very valuable when it becomes possible to launch space probes into orbits designed to make specific observations.

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Standards Division,
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1959 *April* 16.

Note added in proof.—W. M. Kaula (Statistical and Harmonic Analysis of Gravity, *U.S. Army Service Tech. Rep. No.* 24, March 1959) has made an analysis of gravity anomalies assuming that they form a stationary Markov population. He does not derive the lower harmonics from the gravity data but takes them to be determined from satellite observations.

Note added in proof.—Y. Kozai has estimated that $10^6 J_3 = -2{\cdot}20 \pm 0{\cdot}08$. (The Earth's gravitational potential derived from the motion of Satellite $1958\beta_2$ Smithsonian Astrophysical Observatory, *Research in Space Science, Special Report No.* 22, p. 1. March 1959.)

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Geophysics in Pakistan

H. I. S. Thirlaway

Introduction

The Geophysical Institute at Quetta is directed by Mr S. N. Naqvi, Director of the Meteorological Service. It may be said to have been established on 1951 December 26, when the first accurately timed seismograms were recorded, but the specially built laboratories and instrument rooms (the foundations of which were laid in October 1952) were not completely occupied until September 1953. The Institute is 5 600 ft above sea-level at the base of a 10 000 ft peak 5 miles west of Quetta. Laboratory and instrument room space of 12 000 square feet, and 5 000 square feet of library, workshop, service and store rooms, were designed and sited by UNESCO advisers and built by the Pakistan Government on seismic resistant principles enforced by law in Quetta.

Plans for the Institute were initiated in 1950 by Mr Mohammad Aslam, former Director Meteorological Service, and by Dr Malcolm Adiseshiah, Chief of the Technical Assistance Department, UNESCO (now Assistant Director General). The intention was to create a centre in Pakistan for the geophysical study of the Earth and its atmosphere. Atmospheric physics, geodesy, geomagnetism, and seismology were to be the basic fields of activity; geodesy and the absolute field geomagnetic survey being executed by the Survey of Pakistan. The basic agreement between Pakistan and UNESCO was signed in the autumn of 1950. It provided four consultants* in the basic fields, pilot and demonstration equipment to the value of $35 000 and 6 fellowships. Each year since 1951 UNESCO has continued to support the Institute with consultants, equipment and fellowships. The Meteorological Service built and equipped the Geophysical Institute at a cost of about $120 000 and in addition has set up seismological observatories at Karachi, Lahore and Warsak Dam (near Peshawar) in West Pakistan, and Chittagong and Sylhet in East Pakistan. It now plans to build a student hostel at Quetta and a similar Institute at Chittagong to serve East Pakistan.

From 1955 the Survey of Pakistan, directed by Mr M. N. A. Hashmie, expanded its co-operation with the United Nations through another Specialized Agency, UNTAA, to include photogrammetry and the manufacture of survey instruments. Assistance in the geodetic work was then transferred to UNTAA for administrative convenience. New laboratories for the Geodetic Institute of the Survey of Pakistan are expected to be built in Quetta within the next two years. The photogrammetric work of the Survey of Pakistan at Rawalpindi is based on an air photo survey of all West Pakistan (save the high Karakoram region) on a

* E. M. Fournier d'Albe, G. Nørgaard, H. I. S. Thirlaway and K. A. Wienert. With the exception of Thirlaway, all served up to 1954 when responsibility for all observatory work at the Institute was handed over to Pakistan. Thirlaway will leave Pakistan in 1960.

contact scale of 1/40 000 carried out by the Photographic Survey Corporation of Canada under a Colombo Plan project.

The Headquarter offices and laboratories of the Geological Survey, directed by Dr E. R. Gee are also in Quetta, and a bi-lateral project with the International Co-operation Administration of the United States provides consultants from the U.S. Geological Survey, scholarships and equipment to assist in the recent large expansion of the Survey. The Survey has a Geophysical Branch with a staff of 6 geophysicists.

The three Departments are well located geographically for the study of Earth physics, since Quetta is situated in the Baluchistan mountain arc at the centre of the re-entrant structures.

The Geology Department of the University of the Panjab, Lahore, has recently added geophysics to its courses and started systematic field work. Though the Department's senior staff of four are all from UNESCO,* it is hoped that local staff will be available by the end of 1960.

Lahore is also the headquarters for the Groundwater Organization of the Public Works Department. This group are applying geophysics to the serious water logging and salinity problem in the Panjab. A geophysicist of the U.S. Geological Survey† is associated with the programme.

A serious shortage of well-qualified geologists and geophysicists in University and Government service is due partly to the demands of some half-dozen oil companies which are actively prospecting the Indus Basin and East Pakistan. At least four promising geophysicists have left the country to seek better financial and professional opportunities abroad. Before 1951 however there were no geophysical centres, laboratories, nor geophysicists in Pakistan at all. This article summarizes the results in Earth physics since that time and introduces the geophysicists responsible.

Seismology

The central observatory was installed at the Geophysical Institute in 1953 by Mr Moiduddin Ahmad with the assistance of Messrs R. A. Shamsee, A. H. Naqvi and A. Rehman. Mr Moiduddin subsequently installed the Karachi station, and Mr A. Q. Khan those of Lahore, Warsak Dam (Peshawar), Chittagong (new site) and Sylhet. The first station at Chittagong was started by Dr M. A. Choudhury in 1955. Mr A. Q. Khan, the present seismologist, was responsible for calibrating all the stations, and introducing amplitudes and magnitudes into the consolidated monthly bulletin of seismological readings.

Quetta, Karachi, Lahore and Chittagong are equipped with Sprengnether 1 s period sets. Warsak Dam has a single horizontal 1 s period Sprengnether and a vertical Willmore component, and Sylhet a Wenner accelerometer. Quetta and Chittagong are also equipped with Willmore sets, and Quetta with a Wenner accelerometer, a single Milne-Shaw, and two 20 s period Sprengnethers. There is need for a long period vertical to match the horizontal components at Quetta. The maximum drum speed used is 1 mm/s.

The Sprengnether sets have been found very robust, readily adjusted and easily maintained. The Willmore sets, though better for near earthquake work, have proved more difficult to maintain at peak efficiency. They have been used

* K. Helbig from Munich is the geophysicist.
† D. Greenman.

in the field on two occasions with success. The first was after the 1955 February 19 earthquake near Quetta, when the mean epicentre and depth of focus of aftershocks were determined using records from the observatory and three field stations (Khan 1958). Of particular interest is the unusually shallow depth of focus (7 km). This had been suspected to be true of earthquakes in this region on qualitative evidence (West 1935). The sedimentary rocks of the region are generally more than 7 km thick.

The seismicity of Pakistan with special reference to the problem of designing seismic resistant structures has been reviewed (Khan 1956).

At the request of Professor Antonio Marussi of Trieste, who visited the Institute during the course of the Italian Expeditions in the Karakoram region, a review of the seismicity of the Hindukush and Pamir areas was made (Choudhury 1959). The Hindukush is an unusual deep focus (200 km) area, and the question was raised as to whether there is any form of coupling between this area and the adjacent normal focus area of the Pamirs. The epicentral map includes gravity anomalies known at the time. Some new work carried out with Professor Coloumb and Mme Labrouste in Paris is included in the review (Choudhury 1956, 1958). Of special interest is his estimate of 75 km to the Mohorovičić discontinuity in the Hindukush area using a reflected phase off the discontinuity from the deep focus shocks.

Geomagnetism

The observatory was installed at the Geophysical Institute in 1953 by Messrs S. Abu Tahir and S. A. A. Kazmi. The Chittagong equipment was installed for training, in a domestic house rented for the purpose, by Mr S. J. Ahmad, the present geomagnetician. A third station is planned for operation in the far north of West Pakistan, probably at Gilgit.

At Quetta, base values were controlled by means of three QHMs and an Earth Inductor. Askania HTMs arrived last year to replace the QHMs. Ruska and Danish Quick-running magnetographs record the diurnal variations. Full details of the observatory, behaviour of the equipment, and the Annual Summary of observations up to the end of 1955 have been published (Kazmi 1956). The corpuscular eclipse, recorded at Quetta during the total sun eclipse of 1954 June 30, has been described (Kazmi & Wienert 1955). An independent check on the precision of the Quetta Observatory up to 1955 has recently been made (Kazmi 1959) in the course of a study of *Sq* variations under Dr Laursen at Copenhagen.

The ionospheric equipment of the Geomagnetic Observatory is a standard NPL type which was installed by Messrs Ziauddin Ahmad and Musharraf Hussain. A three-mast rhombic aerial with vertical orientation is used. Geomagnetic and ionospheric bulletins are published monthly.

The geomagnetic field survey of the country is being carried out by the Geodetic Branch of the Survey of Pakistan in charge of Mr S. Q. Hasan. Forty new stations were observed in 1953 and 1954 and 3 stations of the former survey repeated by Mr Rajput. Some results are discussed later.

Physical geodesy

Mr. Hasan plans to have a gravity station at the corner of 10-mile squares throughout the country. Each station will be identified by a permanent plinth tied to the first-order level network. In 1955, 104 new stations in West Pakistan

and 67 in East Pakistan were measured with a Worden Geodetic gravity meter by Mr S. I. H. Musvi. The National Base value of gravity used in this work is that measured by Woollard and his colleagues who obtained independent values of 978·9634 cm/s^2 and 978·9626 cm/s^2 at Karachi Civil Airport (Woollard *et al.* 1952). The mean value, 978·9630 cm/s^2, has been adopted. It will be necessary to repeat all the old pendulum stations since some of those already checked are now beyond the desirable limits of precision (see for example Thirlaway 1959). A series of modern pendulum observations between Colombo and Gilgit is desirable for gravity meter calibration in the region.

Both the field geomagnetic and gravity results are due to be published this year in the First Geodetic Report.

Geodynamics

Seismological, geomagnetic, gravity and geological observations in the Quetta re-entrant region are being integrated at the Geophysical Institute.

Earthquake epicentres follow the strike of the Tertiary and the Secondary structures, the great majority being located south and east of a geanticline mapped by geologists of the Photographic Survey Corporation. The relatively few earthquakes north and west of this structure are associated with a series of wrench faults trending NNE. (The largest of these, which can be traced for several hundred miles on air photographs, passes through the towns of Chaman and Nushki.) The formal projection of epicentres along any section normal to the re-entrant structure shows a concentration of seismicity south or east of the geanticline.

The largest earthquake of recent years in the re-entrant is that of 1955 February 18, about 10 miles north of Quetta, at a depth of 7 km (Khan 1958). The first motion recorded at Quetta was dilatation, as were those of all the Indian stations save Madras. Hamburg and Stuttgart recorded compression. Underthrusting or sinistral wrench faulting near Quetta are consistent with the few observations that have been collected.

A feature noted in the Quetta seismograms from some earthquakes within a few miles at Quetta is the unusually large vertical P amplitude. The simplest explanation is that the foci of these earthquakes are almost beneath Quetta. The S–P intervals will therefore measure approximately the depth of focus. This is usually less than 15 km, in agreement with the findings of West and Khan already cited.

Some 50 vertical force values of the Earth's magnetic field were measured along the main Chaman–Sibi road (175 miles) through Quetta in 1953 and 1954 by the Survey of Pakistan. There is a step of about 200γ in the otherwise undisturbed gradient of about 17γ/mile throughout the traverse. This occurs 25 miles south of Quetta at the head of the Bolan Pass leading down across the strike through the re-entrant structures. On the other hand, 12 vertical force values between Quetta and Kalat (80 miles) along the strike show no significant break in the regional gradient.

Gravity anomalies observed by the Survey of India within the re-entrant structures have the following features (see Survey of India Geodetic Reports for the gravity values).

(a) The isostatic isoanomalies follow the geological strike of the mountain arc;

(b) large negative isostatic anomalies (-75 mgal) border the Indus valley boundary of the arc;

(c) the isostatic and free air anomalies increase positively with height (ranges: 200 ft to 8 000 ft, −75 mgal to +45 mgal (isostatic), and −140 mgal to +120 mgal (free air)).

(d) the Bouguer anomalies are independent of height. (Mean value of about −150 mgal over the same altitude range.)

This behaviour is in marked contrast with that of the gravity field in other mountainous areas. In the easterly striking Mekran ranges of similar tectonic age to the south, for example, the expected behaviour of Bouguer and free air anomalies in a compensated region is observed (Thirlaway 1959).

It is believed that the rapid increase in thickness of relatively light Quaternary and Tertiary rocks between the geanticlinal axis and the Indus Basin is the cause of these four unusual features of the gravity field. If the mean density of these rocks is of the order of 2·4 g/cm^3, and not 2·67 g/cm^3 as assumed in the isostatic reductions, a variation in thickness up to a maximum of 30 000 ft of low density rocks will remove most of the unusual features in the gravity field. An oil boring by Pakistan Petroleum Ltd., at Bannh (Sibi) near an isostatic anomaly of −75 mgal ended at a depth of 13 001 ft in Siwalik shales. The figure of 30 000 ft is greater than Glennie's downwarps by a factor of two (Glennie 1956). However, general agreement has been reached on the existence of considerable crustal downwarps bordering the mountain arc of the Quetta re-entrant. Confirmatory seismic reflection work by oil companies is not yet available for release.

The geology of the mountain arcs is everywhere exposed and the more difficult to interpret for that reason. The oldest rocks exposed are Permo-Carboniferous along the line of the inferred geanticline. North and west of this line the sequence thickens rapidly as a shale facies; south and east as a limestone facies. Thrust plane traces are everywhere parallel to the sinuous strike of the arcs. Over-thrusting is everywhere towards the south and east. A complementary series of active wrench faults seem to be causing the deformation of the arc round Quetta. The larger set has been mentioned in connection with the seismicity of the area; recent movement is demonstrable. This sinistral series strikes NNE and the complementary dextral series WNW. A maximum principal stress directed horizontally in a direction S 20° E (or in the complementary direction) could produce both sets of wrench faulting (Anderson 1951) and the effect of continued movement along both sets would be to deform a simple arc into a sinuous re-entrant.

A model which fits the observed seismicity, gravity field, and geology has been constructed. This requires initial failure by thrusting (overthrust to south and east) in a manner identical to the classical island arc development, and subsequent deformation of the simple arc by wrench faulting. The required change of the intermediate principal stress to the vertical direction is brought about in a natural manner by the accumulation of mountain ranges during the initial thrust-type failure. The magnetic anomaly of 200γ also fits this model if it is assumed (contrary to experience) that the thick sediments underlying the overthrust basement slice are slightly more magnetic than the underthrust basement. The position, but not the sign, of the magnetic anomaly was predicted from the model. A means of testing the model is by fault displacement studies.

Applied Geophysics

Oil prospecting by private companies has resulted in large gravity coverage of the lower and middle Indus Basin and East Pakistan, including an off-shore survey

in the Bay of Bengal. Seismic reflection surveys are also extensive. Mr Menan Khan, Superintending Geophysicist of the Geological Survey, has undertaken the collection of gravity information released by the companies.

Gravity and seismic surveys can assess groundwater resources in the western mountain valleys (Ahmad & Thirlaway 1958). It is often possible to distinguish between river and lake environment deposits by locating buried channels with gravity surveys. The lake deposits are normally much less porous and therefore less productive of water. Seismic surveys are used to locate layering within the valleys. The structure of the valleys, bounded by limestone ranges, is rift-like with terminating cross faults. A few thousand feet of light material (presumably mainly Siwalik) has collected within the rifts; in the Quetta valley for example, there is some 4000 ft of relatively light material at its deepest area just north of the city. Maximum anomaly variations between the centre of the valleys and the denser rocky edges reach 20 mgal with gradients of 10 mgal/mile. The density differential is about 0·3 g/cm^3. The gravity work has been extended to cover several adjacent valleys (Mufti & Siddiqui 1959).

Small haematite deposits have been located near Dalbandin with vertical magnetic balances (Farah & Aslam 1958). Four thousand stations were observed on a 200 ft grid. Shear zones were well delineated on this scale. The largest anomalies in the area were 7000γ, and they appear from size, shape and drilling information to date, to be volcanic necks. This was a combined operation by the Geological Survey and Geophysical Institute.

Other work by the geophysical branch of the Geological Survey includes coalfield gravity and seismic surveys, a large number of electrical resistivity surveys for water, and dam site surveys. The Geophysical Institute and University of Lahore have been used on some of these operations. It is hoped to begin publication of results in the near future.

Dr. K. Helbig began systematic magnetic surveys with University of Panjab students over the Hidden Range this year. This structure has already been well explored gravitationally and it is possible that oil company interests may again extend north to the Range. Other geophysical work in this area has been carried out by the PWD Groundwater Organization, a group formed a few years ago to study waterlogging and salinity problems caused by irrigation works. Very detailed electrical resistivity and well logging surveys were introduced by consultants of the International Co-operation Administration. The results are being prepared for publication.

The Geophysical Institute,
Quetta:
1959 *June.*

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Reports of Discussions

Geophysical Discussions of the Royal Astronomical Society

A discussion was held in the rooms of the Royal Astronomical Society on 1958 November 28 at 16h 15m on **Turbidity Currents: High-Speed Underwater Flow of Sediment Suspensions.** It was under the Chairmanship of *Dr G. E. R. Deacon, F.R.S.* He opened the discussion by commenting on the prominence this problem of oceanography had acquired as a consequence of its being recognized that turbidity currents may have been responsible for the breaking of submarine communication cables. The Chairman introduced each of the speakers in turn.

Dr A. S. Laughton of the National Institute of Oceanography dwelt on three classes of evidence of turbidity currents from which our knowledge of them has derived. First, turbidity currents have been observed in lakes. In Lake Mead, formed behind the Hoover Dam on the Colorado River, currents have been observed moving with speeds up to half a knot, one or two metres in thickness, and lasting several days. These currents carry a suspension of silt and clay which can reach a density of 2·05 and they originate in the turbid river which empties into the lake. Second, in the deep oceans turbidity currents are thought to be responsible for the deposition of relatively coarse (sand) layers of sediment far from land. Also, such currents appear responsible for the breaking of submarine cables. Heezen and Ewing calculated the speed of such a current from cable breaks following the Grand Banks earthquake of 1929. It appeared that a current with a speed of 60 knots could have been responsible. From this study Kuenen calculated the size of the current and the quantity of material moved. Subsequent submarine coring in this area of the Atlantic by Ewing and Heezen confirmed the nature of the current through the nature of the sediment it transported. Similar cable breaks due to turbidity currents have been known to occur off the Algerian and Columbian coasts. Third, laboratory experiments can produce turbidity currents. It can be seen that the currents produce lateral as well as vertical variations in sedimentation. Kuenen, after many experiments in tanks and troughs, produced a formula, modified from the hydraulics for the flow of a river, relating the velocity to the depth, slope, and density of the current. The speaker ended by stressing the apparent importance of these currents in producing features of lake and ocean floors: the flatness of the bottoms and the movement there of coarse and poorly sorted materials. He called for more knowledge of their actions to better understand their mechanics.

Dr R. Stoneley, F.R.S. of Cambridge University, spoke next. In discussing the suggestion that the breakages of submarine telegraph cables, following the Grand Banks earthquake of 1929 November 18, occurred through the passage of a turbidity current, he remarked that the hypothesis had the merit of being more free from *ad hoc* assumptions than some of the other explanations proposed. The idea merited further examination from the hydrodynamical standpoint. In particular, two aspects had to be considered, (a) whether the turbidity current would

spread longitudinally too greatly to be effective, (b) whether the motion of the current would persist over distances of the order of 1 000 km without being damped out by viscous forces. Under (a), the spread of the sediment-charged mass of water longitudinally had much in common with the problems of the surge following the breaking of a dam and of the subsidence of a volume of water, considered respectively by R. F. Dressler, and by W. G. Penney and C. K. Thornhill. A rough calculation based on the work of these authors suggested that the depth of the current as envisaged by Kuenen would be much too great; the hypothesis would be more workable if the current had a thickness of only a few metres. The question (b) depended on the viscous drag of a stream at high Reynolds numbers; the theoretical and experimental work of G. H. Keulegan on arrested saline wedges suggested a much lower drag coefficient than was appropriate for lower speeds. Keulegan's experiments showed that the logarithm of the drag coefficient depended linearly on the logarithm of the Reynolds number, and an extrapolation of this result indicated a thinner turbidity current, of vertical thickness about 5 m. Such a thickness would help to meet the difficulty mentioned under (a). Dr Stoneley said that further investigations of these hydrodynamical problems were very desirable, and examination was needed of the effectiveness of such a current in transporting and sorting sediments to the degree required.

Dr T. H. Ellison of Manchester University was the final speaker in the formal portion of the discussion. He dealt with a mathematical consideration of the mixing of the turbulent flow with the clear water above thus diluting it and thickening the flow. Due to this thickening any filaments of the dense flow which entered the upper layer would be reincorporated and not escape from the current. The problem was to know when this dilution would halt the flow. An experiment by Turner consisting of injecting a two dimensional jet of fresh water into salt water had been analysed and it gave some estimates of the Richardson number. If this was low and combined with a high slope there would be much mixing. But with the small slopes occurring in the case of the Grand Banks turbidity current the Richardson number was large. Using the figures of Kuenen the speaker showed the distances and velocities of that current to be reasonable. The development presented had derived from a study of the migration of methane along the roofs of mine tunnels.

In the general discussion which followed *Dr K. E. Latimer* made some observations of concern to submarine cable engineers. A layer of sand one metre thick could be enough to bury a caple beyond the depth at which it might be recovered by a grapnel. A 3 mile length of cable with little slack might be broken by a current of 2–3 knots. Thus turbidity currents with velocities and densities as great as those mentioned in the discussion could not only break but sweep away the cable. Turbidity currents were also of interest in the matter of understanding the origin of submarine canyons. Comm. Cousteau had observed a turbidity current started accidentally. The material involved in the Grand Banks turbidity current had presumably come from the St. Lawrence River. This river provided an excellent example of entrained flow for sea water amounting to 100 times the river's volume was carried 500 miles in a current of about 2 knots. *Dr Latimer* asked if such entrainment might explain sudden temperature variations observed by submarine cable engineers. He also asked if turbidity currents seemed to be acted on by the Coriolis force.

Dr Strong described how in decanting, a more rapid separation of fluid and sediment could be effected by tipping the container to increase the surface and

permit flow of the dense lower layer along the sloping side of the vessel. Such a continuous rather than an avalanche current was comparable to the difference between the currents discussed for the ocean floor and those formed where a river entered a sea. Snow avalanches had many similarities to the former type.

In replying to Latimer *Dr Laughton* said that one difficulty in the problem of submarine canyons was to know what started them. A fault or fault zone seemed a possible explanation. Regarding the importance of the Coriolis force he observed that there was a tendency off the Pacific coast of North America for currents to turn left rather than right as they came out of submarine canyons. He said Mennard held the view that this was due to deposition causing a levee to be built up on the right-hand side. Latimer confirmed this observation.

In a series of short comments concerning the breaking of submarine cables points stressed included that the tension in the cable was also partly due to the motion of the ship laying or grappling the cable and that a current acting over a long length of cable could produce a large tension. Laughton stressed that the central region of a turbidity current was one of erosion. Strong said that small movement in sediments could shear a steel tube. Mention was made of man-made currents following bomb explosions in a harbour.

The Chairman asked Ellison if there was a layer of mixing in front of a snow avalanche. The reply was that while the speaker had not looked into the problem he wondered if the particles were not too dense for this to be so.

The Chairman thanked the speakers and other contributors to the discussion and adjourned the meeting at 18h 05m.

A discussion was held in the rooms of the Royal Astronomical Society on 1959 January 23 at 14h 15m on **The Automatic Reduction of Geophysical Data,** under the Chairmanship of *Sir Edward Bullard, F.R.S.*

The Chairman began the discussion by answering the possible question of why one would wish for the automatic reduction of data, by saying that this was because of the size of the problems undertaken and the heavy burden of reducing some data by desk calculations. It gave greater accuracy and saved time for the scientist. He gave as an example his application of automatic reduction to magnetometer data from marine traverses obtained by *Discovery II.* This method illustrated steps typical of many operations. In obtaining the data there was a chain consisting of the measuring equipment, an analogue-digital converter, a punch control and the punch. A chart recorder to plot the data for visual monitoring of the measurements was very helpful, as was the automatic input of additional digital data through the punch control. The speaker preferred to punch on to a tape because the order was kept clear. There was no shrinking to cause trouble in the automatic reading of the tape as was sometimes the case with cards and one number was represented by one row of holes. In the automatic reduction process carried out on the EDSAC II machine, the date was first printed on the output, then the time was read, the magnetometer data read, the magnetic field calculated, heading and diurnal corrections applied, differences between the value determined and

neighbouring values tested to reject erroneous observations, the ship's position was interpolated and the output of the machine punched on tape. It was stressed that the punching of indicators on the tape could help by telling the machine what was to follow and by providing for contingencies. The process of testing figures with some limits of variation was very useful for removing faulty observations and if these were printed separately, later inspection could ensure that valid data were not discarded.

The Chairman then introduced the speakers and recommended that the discussion of each paper should immediately follow its delivery.

Mr R. H. Merson of the Royal Aircraft Establishment spoke on the reduction of interferometer and kinetheodolite observations of Earth satellites. This work had been in progress since shortly after the launching of the first Russian satellite. Prediction of the position of the satellites had been of first concern but the study of their orbits was later extended to obtain information about the air density at high altitudes and about the gravitational field of the Earth. Various optical and radio methods of observation had been used; visual observation, radio Doppler studies, radio-interferometry, pilot balloon theodolite observations and kinetheodolite observations. The last method gave an accuracy of $\pm 0{\cdot}005$ deg in angle measurements and of $\pm 0{\cdot}02$ s in time. The less accurate methods were still useful in obtaining initial estimates of the unknown elements of the satellite orbit. This orbit was then adjusted by successive approximations to give a least-squares fit with all the observational information. Automatic reduction of the data in a computer was necessary to achieve the speed needed in a prediction service. A Pegasus digital computer was used for the major part of the work. From an interferometer record a table of two shifted direction-cosines against time was obtained; the shifted direction-cosines were true direction-cosines plus unknown constants. These data were used to find the height, longitude and time as the satellite crossed the observer's latitude, and the orbital inclination. In making this analysis the size and shape of the orbit (represented by the semi-major axis a and eccentricity e) were assumed to be known. a and e were obtained from the nodal period T and perigee distance r_p. T was provided by the prediction service and r_p, which changes comparatively slowly, from a combination of various types of observation. Initial estimates of the unknown elements were given to the computer which then worked out the values of the shifted direction-cosines in terms of these estimates, compared the results with the observed values, and used the residuals to improve the estimates by the method of differential correction. Kinetheodolite observations (elevation, azimuth, time) were converted to direction-cosines and the analysis then proceeded along the same lines as indicated above.

A later computer programme transformed the results into standard orbital elements. The behaviour of one of these, the Right Ascension of the ascending node Ω, had been used to estimate the values of coefficients in the expression for the Earth's gravitational potential. From observations of Sputnik 2 (1957β), combined with American results from their satellites 1958α and 1958β2, it had been deduced that

$$J = (1{\cdot}625 \pm 0{\cdot}001) \times 10^{-3};$$

$$D = (8 \pm 3) \times 10^{-6}.$$

It followed that the Earth's polar flattening $e = 1/(298{\cdot}15 \pm 0{\cdot}09)$.

During the discussion the *Chairman* asked if this slightly higher value of the

ellipticity over older results might be due to pear-shaped harmonics of the gravitational field or a poor distribution of the earlier results. *Merson* said he thought the latter seemed likely. *Dr A. H. Cook* suggested there might be poor statistical reduction of the older figures.

The second speaker was *Mr G. A. Corby* of the Meteorological Office who discussed numerical weather forecasting with the aid of computers. The forecasting proceeded from the meteorological data recorded at various weather stations and compiled on maps. These data could have been selected from maps for computing by human agency but it had been thought more desirable to let the computing machine enter into even this part of the work. The machine took the data and interpolated between the observed points to obtain values of the variables at a gridwork of points evenly spaced over the area covered. A least squares reduction was then begun to minimize the difference between observed and calculated values with a weighting of points to favour data from nearby meteorological stations. Some maps were shown which compared favourably the machine controlled maps with the hand-drawn ones. This project was still in the development stage.

Questions were put to the speaker about transmission errors. He replied that there were two or three errors per 250 observations (the number of weather stations serving western Europe). *Mr E. Gold* said that when line telegraphy had been in use for transmitting observations, about one figure group in 2000 had been the frequency of transmission errors. This order of frequency for errors seemed inevitable and demanded a well-distributed network of observing stations for proper detection. He asked how long it took to prepare these charts with the aid of the computer. The speaker replied that the time from beginning to end was one and a half hours. *Dr Fox* asked whether local irregularities in the observed data were not extremely troublesome as the numerical processes for solving equations appropriate to a selected model of the atmosphere could be somewhat unstable. *Mr Corby* replied that short-period disturbances could be troublesome to forecasting and a smoothed curve was most desirable.

Dr M. H. P. Bott of Durham University spoke next about his use of a digital computer to reduce and interpret data about the gravitational field obtained by regional surveys over geological formations. One application had been to compute the correction due to the terrain of the area. Another was to calculate the gravitational field over irregularly-shaped bodies in the Earth's crust. When these geological units could be divided into a set of prisms, or two-dimensional bodies whose boundaries were straight lines, the problem was attacked by relaxation methods. Assuming the density contrast between the body and the surrounding rocks to be known, the shape of the lower surface of the body could be calculated if the top surface was defined, by successive approximations. In the discussion which followed this paper the point was made that this knowledge about the density contrast was necessary if there were not to be multiple answers.

Dr D. C. Martin, from the Royal Society, outlined the problems of data processing arising from the International Geophysical Year. Three regional World Data Centres were collecting the data from each subject of study. Some of this was to be published in the *Annals of the I.G.Y.*, the rest would appear in established publications. Automatic treatment of the data was not planned for the whole of it but doubtless much would be handled this way by the investigators concerned with it. *The Chairman* pointed out that a general treatment of all the data by automatic methods had been discussed and a decision taken against it by the British National Committee for the I.G.Y.

The fifth speaker that afternoon was *Mr M. J. Tucker* of the National Institute of Oceanography. His remarks referred to measuring and analysing the spectrum of random-gaussian processes of which sea waves and turbulence were typical examples. The basic limitation, independent of instrumentation, was expressed by the formula

$$e^2BL = 1.$$

Here, e was the proportional random error in the estimate of $E(f)$, the power density of the spectrum; B was the bandwidth of the equivalent filter, the range of frequencies over which $E(f)$ was smoothed and L was the length of the record or the time of observation. For a 10 per cent accuracy and 10 per cent frequency resolution, this required a record containing 1 000 waves: a digital record might need to consist of 5 000 numbers. In a naturally occurring phenomenon, the characteristics might have changed over a period of time of this length. By comparing two related records (for example, surface wave height and the pressure on the sea bed immediately below), the random errors might be reduced or eliminated and higher accuracies be obtained.

The speaker said analogue analysers were generally expensive to buy and cheap to run, whereas if a digital computer could be hired, digital methods involved little capital cost but a high cost per analysis. Digital methods were capable of greater accuracy. Digital methods that used correlation as the first step were economical of computer time and best for many purposes, though periodogram analysis and numerical filters could be useful in particular applications.

In the discussion which followed this paper the point was raised that in some cases a natural phenomenon was stable enough for a sufficient length of time to justify the data being recorded as digits. *The Chairman* asked if it was not true that with digital Fourier analysis the time requirements would be similar to those of analogue methods. The speaker replied that the digital analysis required many more multiplication operations and was thus likely to need more time. *Dr P. Vigoureux* pointed out that for a disturbance which had appreciable periodic components a spectral representation could be obtained only by using a very narrow sampling band. In this case the measurement was no simpler than were autocorrelation methods. He stressed that the sampling band width should always be stated for such measurements.

The possible use of a digital computer for ionospheric forecasting was the topic of a communication by *Mr C. M. Minnis* of the Radio Research Station. Such forecasting was concerned with making estimates of the highest radio frequencies which could be used in transmitting radio waves for long distances by reflecting the waves from the ionosphere. The first stage was to study records of certain ionospheric measurements and then, from a given trend of solar activity, predict future values of ionospheric parameters. The second stage was to compute the required radio frequencies which were functionally dependent on these parameters. As an illustration of how a digital computer could help with these tasks, forecasting the critical frequency (f) of the $F2$ layer was described. Observation showed that the monthly median values of f were useful and that they depended on latitude and longitude as well as on the intensity of solar radiation, hour angle (H), and the mean solar declination (D). The solar intensity could be represented by a linear relation between the critical frequency and the ionospheric index (I_{F2}). By selecting values of H and D the solar intensity was expressed as a function of geographical location. Thus a large mass of hourly observations of f was

reduced to about 1 000 values (for a given month, from 65 locations) which could be dealt with by a computer at one time. The machine could then interpolate the functional dependence of f on I_{F2} for any location and, given a forecast value of I_{F2}, compute a forecast value for f. In this way the derivation of the highest frequency which could be used for radio communication between two stations had been reduced to the evaluation of certain mathematical relations at a series of points along the great circle between these stations.

Mr A. T. Dennison, British Petroleum Company, followed to speak about the use of digital computers in exploration geophysics. Applied to surveys of the gravity or magnetic fields, there were many operations of routine treatment of the data obtained by these surveys, similar to those described by Dr Bott, which could be treated by automatic reduction methods. Large storage capacity was perhaps the most restricting requirement for these computer applications. In seismic prospecting, a digital presentation of the data was not favoured. Here automatic treatment was usually concerned with corrections for one record at a time. It might be possible to compute automatically the correction required for the effect of the weathered layer on a seismic reflection from the records of first arrivals of the reflected waves. The corrected records were then combined for a visual presentation as profiles, variable intensity strips, or in some other manner. On the combined records a correlation from trace to trace was made to construct an interpretation of the subsurface structures. The judgment of which arrivals should be chosen for correlation was preferably left to an interpreter. Magnetic recording of the seismic traces was a relatively recent innovation to these methods. It permitted a greater range of frequencies to be recorded and in the subsequent automatic reduction of these data, various filters could be introduced.

The eighth speaker was *Dr G. W. Lennon* of the Liverpool Observatory and Tidal Institute. He spoke on the processing of Earth tide data. This aimed at measuring the yield of the solid Earth to the gravitational attraction of the Sun and Moon and thereby obtaining information about the Earth's elasticity. This study still required work on instrumentation as well as data-processing techniques. Records extending over long periods were difficult to obtain because of discontinuities. Many spurious contributions were difficult to separate from the tidal effects on the record. These could have a period very near that of the tide, as, for example, meteorological effects. An accuracy of 1 per cent was required in the basic values for the geophysical interpretation to proceed. In the work at the speaker's Institute a modification of the methods developed by Dr Doodson was used to analyse 29 days' observations. This was a manual technique involving numerical filters to separate the tidal species, and correction of the individual constituents. Four reasons were given why digital computer techniques were unsuited to this work: (1) there was not a large amount of data but rather a collection of small units; (2) the elimination of effects due to non-periodic movements had to be done at each stage of the analytical process; (3) in considering labour economy the analysis was the least demanding stage of the work; (4) it was desirable to preserve the non-periodic effects recorded for further geophysical investigations.

The final speaker was *Dr H. M. Iyer* of India, temporarily at the National Institute of Oceanography. In the analysis of microseisms a three component seismograph station should provide information about the amplitude, period, spectra, direction of arrival, and composition of the impulses received. Dr Iyer described an automatic analyser which had been designed and applied to an

electronic feedback seismograph at the National Institute of Oceanography to do this analysis directly from the seismograph signals. The output of a seismograph was fed into a high torque galvanometer fitted with three systems of contacts. These contacts were used (1) to examine the position of the seismograph arm at short intervals, (2) to count the transits of the moving arm from the positive to the negative side during a known length of time, and (3) to examine whether two sets of waves were on the same, or different, sides of the zero line during a known length of time. This last technique was one due to Tomada and used for calculating the correlation coefficients between the different seismic components in order to analyse the composition and direction of arrival of the microseismic waves. The speaker stressed that a treatment of the data such as this, which automated part of the analysis with simple equipment, could be of great help to scientists in countries such as his where the resources often did not permit more elaborate automation.

The Chairman thanked the speakers and other contributors to the discussion and adjourned the meeting at 18h 20m.

A discussion of papers received during the session was held in the rooms of the Royal Astronomical Society on 1959 March 20 at 16h 15m under the chairmanship of *Dr R. Stoneley, F.R.S.*

The first speaker, *Dr D. W. Allan*, presented the paper by Mme H. A. Lubimova on *The Thermal History of the Earth* with Consideration of the Variable Thermal Conductivity of its Mantle (1958). Mme Lubimova had made an important contribution to studies of the Earth's thermal history by making numerical calculations which allowed for a thermal conductivity varying both with depth and with temperature (thus taking into account radiative transfer as well as normal thermal conduction). She found that the thermal history was still largely dependent on the initial temperature, the age of the Earth, and the distribution of radioactivity assumed, as with simpler mathematical models using constant conductivity. The course she proposed for the Earth's thermal history was, very briefly: the Earth was formed by accretion of particles about 5 billion years ago. It was originally cold but due to radioactive sources, kinetic energy of colliding particles and pressure, it heated up. Differentiation of radioactive materials was continuous due to partial melting but main differentiation occurred 1·6 billion years after its genesis. The increase of temperature was more rapid at greater depth than closer to the surface, but the core which was assumed as a mixture of iron and silicates started to melt only $1\text{–}2\times10^9$ years ago and is still melting.

Due to a thermal conductivity minimum at a depth of 50–100 km heat generated in the centre could not escape. The depth of the region of escape was about 500 km. The temperature inside this region had decreased about 100° C in the last billion years. The Earth as a whole could not melt in the process of its evolution.

Another possible thermal history for the Earth, which J. A. Jacobs and D. W. Allan discussed, assumed a slightly older Earth which was almost completely molten at $4{\cdot}5\times10^9$ years ago causing differentiation of radioactive materials.

There was partial remelting near the surface later. The core was always molten and the inner core was formed at the beginning of the process of solidification.

It was interesting that, on both these models, the Earth was heating up rather than cooling below 700 or 800 km depth. Might this be an explanation of the cessation of deep-focus earthquakes at that depth?

Evaluating mathematical theories available at present, *Dr Allan* said that they had reached a necessary degree of complexity. However, more experimental work was needed, particularly on radioactive abundancies. More knowledge of the initial conditions of the solar system would be very helpful. It was imperative to decide how long ago the solar system was evolved and how long it took for the formation of the planets. In the discussion which followed *Dr Runcorn* said that he had been working recently on the mechanism of continental drift and that a continuous differentiation of radioactive material would help. For a model study, one must take into account convection currents in the core, and gravitation. *Dr Benfield* said that the surface of the Moon had not changed much and questioned whether it could be used to explain thermal history. *Dr Allan* answered that, according to his model, the Moon was solid throughout and cooled all the time. *Dr A. H. Cook* noticed that the curve of the melting point temperature near the surface would allow for the presence of magma for volcanoes. *Dr Stoneley* mentioned that nobody attributed any anomalies on the seismograms to pockets of molten lava.

The second paper presented was a joint paper by *Mr L. R. Cooper* and *Mr C. Jones* on *The Determination of Virgin Strata Temperatures from Observations in Deep Survey Boreholes* (1959). The knowledge of vertical temperature distribution in the ground around a favourable geological formation was of great importance in designing a new mine installation, a ventilation system and in evaluating future operational efficiency. The difficulty in obtaining temperature distribution was due to the fact that, once the ground had been disturbed, the virgin temperature changed. In order to measure the virgin temperature a special technique had to be developed including a specially designed thermal probe which could be used in conjunction with the drilling rig. The probe used a thermistor sensing device which was connected by a two-core cable to a Wheatstone bridge. The main advantage in using a thermistor probe was that a thermistor had an impedance of about 400 kΩ so that small variations in cable resistance due to changes in temperature along the borehole could be neglected. Some early measurements in coal measure rocks indicated that temperature geothermic gradient changes from 100 to 40 ft/°F. This was attributed to some low conductivity coal pieces in the strata pierced by the borehole. In the discussion, *Dr T. F. Gaskell* asked about the magnitude of the correction which had to be applied to measured temperatures in order to obtain true virgin temperature. *Mr Cooper* answered that the correction was 1 to 4 or 5 °F, proportional to depth and depending upon the time necessary for withdrawal of the drilling rods. *Dr W. H. Bullerwell* mentioned that the Geological Survey sometimes performed similar measurements but due to operational procedure it was not possible so far to pay much attention to the temperature of the bottom of the hole. *Dr A. H. Cook* suggested that the techniques described in this paper might be applied to heat flow measurements.

The third paper of the session was presented by *Dr A. H. Cook* on *Determinations of the Earth's Gravitational Potential from Observations of Artificial Satellites* (1958). If the Earth's gravitational potential consisted of $1/r^2$ terms only, the orbit of any artificial satellite would be a plane ellipse with focus in the centre of

the Earth. The fact that the satellite orbit rotated around the major axis was due to higher order terms of the potential. By observing the retrograde motion of the orbital plane the values of higher terms in the expression for potential could be evaluated. Utilizing observations of the Vanguard satellite, the following values had been obtained by Mr R. H. Merson:

$$10^3 J = 1{\cdot}625 \pm 0{\cdot}001$$
$$10^6 D = 8 \pm 3$$

and for the ellipticity $e^{-1} = 298{\cdot}2$.

Dr J. de Graaff-Hunter pointed out that the coefficients J and D, as found in geodetic work including gravity, were evaluated on the assumption that the level surface was a spheroid and D was chosen to fit the measurements. The spheroidal shape was worked out on the assumption that the Earth was a fluid rotating body and consequently we should not hang on to the old value of D when the new direct determinations of this factor gave a significantly different value.

Dr R. L. G. Gilbert could not be present and his paper on *An Investigation into the Calibration of Gravity Meters* (1958) was presented by *Dr W. H. Bullerwell.* This paper described the work of Dr Gilbert at the Dominion Observatory, Ottawa. In large scale surveying with gravity meters it was very important to check if calibration constants of gravity meters changed. The repeated tests of two North American and one Worden gravity meter showed that the calibration constant changed over a period of time. It was concluded that measurements over the range of 1000 mgal might be preferable with pendulums. A calibration device built into the gravity meter which resembled the Askania gravity meter construction did not operate as satisfactorily as was expected. Dr Bullerwell then described results of calibration tests done by the Geological Survey over a British calibration line. Six Worden instruments were tested. It was discovered that the change of calibration constant was proportional to the age of the instrument, increasing with time. This could be explained either by the ageing of the instrument or by a change in calibration procedure by the manufacturer. *Dr Cook* pointed out that an accuracy in positioning the ball of 1 part in 10^8 was necessary in order to have a reliable built-in calibration device.

The Chairman thanked the speakers and other contributors and adjourned the meeting at 18h 10m.

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Book Reviews

The Magneto-Ionic Theory and its Applications to the Ionosphere

J. A. Ratcliffe

(Cambridge University Press, 1959, 40s.)

Mr J. A. Ratcliffe is famous, both nationally and internationally, for his capacity to select and simplify the fundamental features of complex phenomena, and his influence has stimulated the Cavendish Laboratory ionospheric group until it has become the Mecca of young research workers in this field. He has taken an active interest in the magneto-ionospheric theory since it was first developed and published one of the first detailed accounts of the results of the theory (*Wireless Engineer* 1933, **10**, 354.) Even today, more than twenty-five years later, this paper can be recommended as an accurate and clear introduction to the subject. Throughout this period, Mr Ratcliffe has been particularly interested in the physical significance of the mathematics and has devoted much time and attention to its demonstration. We may thus expect that his monograph on magneto-ionic theory will not only be a first-class contribution, but is likely to be remarkably clear and readable. Our expectations are not disappointed.

The first impression given by this book is that it could well be taken as a model for similar works in other fields where progress is hampered by the difficulties and complexities of the basic mathematical theory. This is primarily due to the care with which the subject matter and degree of development have been limited so that the important features of the theory are brought out in simple form. At the same time, the intricacies of the theory are investigated sufficiently fully to remove the main difficulties of those who wish to proceed to a more exact and complete analysis of particular problems. The annotated bibliography is designed to guide this development and, in effect, adds several hundred pages of detailed analyses. Great care has been taken to underline the physical significance of the theory, the technique of developing the theory from both macroscopic and microscopic points of view being particularly effective. However, possibly the most valuable feature of the book, both to undergraduate, young research worker, and expert is the collection and clarification of many difficulties and fallacies which have been encountered in the past but are not readily available in the literature.

The author's experience of the difficulties met by research students in understanding and using magneto-ionic theory is probably unequalled and has provoked him to develop new methods of explaining these difficulties. This is reflected in the book, which contains a large amount of valuable material which has never been published before. It is remarkable how little the standard figures and curves usually reproduced elsewhere contribute to the author's argument. The bibliography is arranged so that additional material can readily be found if needed.

Roughly half the book is concerned with the development, properties and physical significance of the magneto-ionic theory and roughly half with the problems

which arise when it is applied to the ionosphere. While, of course, much of this theory can be applied directly to non-terrestrial magneto-ionic media, such as the solar atmosphere, the development is primarily slanted to be directly applicable to the ionospheric problems of the interaction of a radio wave and the ionospheric media. In both cases the analysis corresponds to a ray treatment, the more complex phenomena associated with coupling, which need a full wave treatment, being discussed in outline but not in detail. References are given to the fuller treatments. With this limitation, which is imposed by the author's decision to make his book simple and readable, the book is sound and complete, the only omission noted by the reviewer being the absence of any comment on, or reference to, Booker's (*J.I.E.E.* III 1947, 94, p. 171) analysis of propagation in terms of impedances. While, admittedly, this approach is fundamentally different to that selected by the author and has still to be expanded to be fully applicable to magneto-ionic topics, it provides yet another useful viewpoint the main points of which could have been summarized in an appendix. This would have been particularly valuable to radio propagation engineers, many of whom approach practical propagation problems in terms of telegraph theory. However, this attractive and detailed account of magneto-ionic phenomena in the ionosphere will undoubtedly remove many misconceptions. An excellent summary of the scope of the monograph can be obtained quickly from the list of contents though some readers may find certain of the last few sections to be slightly over-compressed.

This is an exceptional book which can confidently be expected to become a standard text book on magneto-ionic theory as applied to the ionosphere. While, in the future, it may be desirable to add new sections, the main content of the book is not likely to date. There is no doubt that workers in this field will owe a considerable debt of gratitude to Mr Ratcliffe for this valuable monograph.

W. R. PIGGOTT.

Fonctions Sphériques de Legendre et Fonctions Sphéroïdales

Louis Robin

(Tome I 1957, 201 pp. + xxxv 4,000 FF. Tome II 1958, 384 pp. + viii 5,000 FF. Gauthier-Villars, Paris)

The author of these two volumes, and a third which is not yet published, has set out to provide a comprehensive and up-to-date reference book on the properties of Legendre Functions. The first volume begins with a useful detailed summary, under the heading "Introduction et vue d'ensemble", of the contents of all three volumes.

Chapter I starts with the Legendre equation and the author proceeds to derive properties of the solutions $P_n(\mu)$ and $Q_n(\mu)$ for positive integral values of n and real or complex μ. Chapter II extends this analysis to the associated Legendre equation and the associated Legendre functions $P_n^m(\mu)$ and $Q_n^m(\mu)$ for integral values of m and n, n being positive.

Chapter III deals with integral spherical harmonics. Volume I ends with tables giving $P_n(\mu)$, $Q_n(\mu)$, $P_n^m(\mu)$ and $Q_n^m(\mu)$ as functions of μ for small integral values of m and n.

In Volume II the scope is greatly enlarged by extensions to complex values of m and n. In Chapter IV $P_n^m(\mu)$ and $Q_n^m(\mu)$ are defined by means of contour integrals for any real or complex m, n and μ. The properties of these generalized functions are then developed and exhibited as extensions of those for integral m, n. In Chapter V are obtained various asymptotic series for P_n^m and Q_n^m valid for large $|m|$ or $|n|$. Chapter VI deals with the expression of a given function as a series in terms of associated Legendre functions. Volume III will deal with theorems of addition of Legendre functions, applications to ellipsoids of revolution, Gegenbauer functions and spheroidal functions.

These volumes are written with great skill and clarity; they form a compendium of information comparable with Watson's *Bessel Functions*. In general full proofs are given, and the reader is assumed to be familiar with hypergeometric series, the method of steepest descents, asymptotic expansions and Lebesgue integrals. But for anyone needing only to extract a formula suitable for application in a problem of mathematical physics, the summary in the Introduction will form a convenient pointer, and the careful setting out of all formulae with stated conditions of validity should ensure correct use. The author is an engineer and the book arose from needs discovered in dealing with problems of fluid mechanics, geophysics and electromagnetism. Workers in such fields are likely to find it of very great value.

E.R.L.

Russian translations of summaries of original papers

(Russian summaries of the foregoing papers are unavoidably held over until the next issue.)

(ПЕРЕВОДЫ РЕЗЮМЕ ПРЕДЫДУЩИХ ДОКЛАДОВ НА РУССКИЙ ЯЗЫК НЕИЗБЕЖНО ЗАДЕРЖИВАЮТСЯ ДО СЛЕДУЮЩЕГО ВЫПУСКА.)

Geophysical Journal

RECOMMENDATIONS TO AUTHORS

1. Authors are invited to follow the general suggestions for the preparation of scientific papers given in *Notes on the preparation of papers to be communicated to the Royal Society* and in *The printing of mathematics* by T. W. Chaundy, P. R. Barrett and Charles Batey (Oxford University Press, 1954). In particular the abbreviations for the names of periodicals used in the Geophysical Journal are those given in *Notes on the preparation of papers . . .* and the symbols, signs and abbreviations are those recommended by British scientific publications by the Symbols Committee of the Royal Society.

2. Papers should be typewritten, with double spacing and on one side of the paper only. Two copies, one of them on tough paper, should be submitted. All pages should be numbered consecutively. Any necessary corrections may be supplied by the author or the Editor.

3. Tables should be numbered in order. They should be constructed carefully with double spacing between columns and lines. Units should be placed at the head of the column. Tables should be typed on separate sheets and their position in the text indicated in the margin.

4. Illustrations, numbered consecutively, should be kept to a minimum, and the cost of publishing coloured illustrations is usually prohibitive. Figures should be drawn in black ink on white paper. The author should indicate the size of figure required in publication. Lettering should be sufficiently large to allow reduction to the size of the figure. Figures should be numbered in order and referred to in the text. Captions, which must be supplied in addition to the original illustrations, should be typed on a separate sheet. Authors should indicate on the margin of the typescript the position in the text where each figure is to be inserted. Photographs should be submitted as glossy prints, mounted on thick paper, and should be of a size suitable for reproduction. Authors should indicate the size of figure required in publication. Lettering should be sufficiently large to allow reduction to the size of the figure. Figures should be numbered in order and referred to in the text. Captions, which must be supplied in addition to the original illustrations, should be typed on a separate sheet. Authors should indicate on the margin of the typescript the position in the text where each figure is to be inserted. Photographs should be submitted as glossy prints, mounted on thick paper, and should be of a size suitable for reproduction. Authors should indicate the size of figure required in publication.

5. [illegible] should be arranged in the following order: author's name and address, abstract, introduction, [illegible] conclusion, appendix, acknowledgements, bibliography. Footnotes should be avoided as far as possible.

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In Volume II the scope is greatly enlarged by extensions to complex values of m and n. In Chapter IV $P_n^m(\mu)$ and $Q_n^m(\mu)$ are defined by means of contour integrals for any real or complex m, n and μ. The properties of these generalized functions are then developed and exhibited as extensions of those for integral m, n. In Chapter V are obtained various asymptotic series for P_n^m and Q_n^m valid for large $|m|$ or $|n|$. Chapter VI deals with the expression of a given function as a series in terms of associated Legendre functions. Volume III will deal with theorems of addition of Legendre functions, applications to ellipsoids of revolution, Gegenbauer functions and spheroidal functions.

These volumes are written with great skill and clarity; they form a compendium of information comparable with Watson's *Bessel Functions*. In general full proofs are given, and the reader is assumed to be familiar with hypergeometric series, the method of steepest descents, asymptotic expansions and Lebesgue integrals. But for anyone needing only to extract a formula suitable for application in a problem of mathematical physics, the summary in the Introduction will form a convenient pointer, and the careful setting out of all formulae with stated conditions of validity should ensure correct use. The author is an engineer and the book arose from needs discovered in dealing with problems of fluid mechanics, geophysics and electromagnetism. Workers in such fields are likely to find it of very great value.

E.R.L.

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2. Papers should be typewritten with double spacing and on one side of the paper only. Two copies, one of them the top copy, should be submitted. All papers must be accompanied by a summary; summaries are not required for *Letters to the Editors*.

3. Tables are printed without rules. They should be numbered serially with Arabic numerals. Table headings should be brief. Units should be placed at the head of the column. Tables should be typed on separate sheets and their positions in the text indicated on the copy.

4. Illustrations, especially photographs, should be kept to a minimum, and the same information should not in general be given in both tables and illustrations. Line drawings should be in dense black ink on smooth white board or transparent tracing film. Original drawings must not be lettered; lettering should be indicated on copies or photoprints which must be supplied in addition to the originals. Wherever possible, lettering should be kept outside the diagram so that it may be set in type. Diagrams should be drawn at twice to three times the size at which they will be printed. The maximum dimensions for diagrams, when reduced and with their lettering and captions, are 8 in by 5 in. A wide variety of flat tints of dots, lines and shadings can be applied to line drawings by the blockmaker. Photographs for reproduction should be unmounted glossy prints and should be accompanied by lettered prints.

5. References should be quoted in the form recommended by the Royal Society but authors may also quote titles of papers if they wish. References in the text are made in the form (Smith 1940).

6. Authors whose work involves much mathematics should follow carefully the recommendations in *The Printing of Mathematics*.

7. The Editors will be glad to advise authors on any special points of difficulty arising in the preparation of papers for the *Journal*.

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